CDO Analytic Pricing with Subordinator Levy Marshall - Olkin Correlation

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CDO Tranche Analytic Pricing with Subordinator Lévy Marshall-Olkin Correlation

Stefano Giovannitti *

1 Abstract

In this paper we present two (semi)-analytic synthetic CDO tranche pricing formulas using a subordinator Lévy Marshall-Olkin credit correlation model. These formulas can be easily evaluated in terms of machine computational time, therefore they are particularly suitable for the correlation model calibration. To compute the first pricing formula, we assume that the recovery rate and the survival probability is the same for each name of the CDO pool (homogeneous pool approximation). We derive a second pricing formula, under the additional assumption that the number of names in the pool is infinite (large homogeneous pool approximation). Both the two formulas are (semi)-analytic and hold for every class of subordinator Lévy process. Finally, the computational cost does not increase with the number of subordinators.

2 Introduction

In recent years, after the 2008 financial crisis, credit derivatives market has experienced a huge expansion. Both single-name and multi-name products have been actively traded. The most liquid single-name products are credit default swaps (CDS), which guarantee protection against the default of a credit name. Some of the most liquid multi-name products are credit indexes, first or n-th to default swaps and credit default obligation (CDO) tranches. Due to the great impact of correlation in multi-name credit derivatives, financial industry has developed an increasing interest in credit correlation modeling.

An industry common approach to credit correlation modeling is based on copula functions [11]. Among all the copula functions, the most famous one is the one-factor Gaussian copula [9], [13]. It is specified by a single constant parameter representing the pairwise correlation. The Gaussian copula is widely used because it is particularly tractable from an analytic point of view. Moreover, the fact that it is specified by only one parameter means that the model calibration does not require many market data inputs.

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However, Gaussian copula has some drawbacks when employed in practical cases. The first one is that it underestimates default correlation in case of extreme events. The reason is that it shows asymptotic independence. Another drawback is that it is not a time-dependent correlation model. Finally, CDO tranche market quotes exhibit a phenomenon called correlation-skew, which, in terms of Gaussian copula correlation, means that different tranches are priced with different Gaussian copula parameters. Detailed discussions on the limits of the Gaussian copula for CDO pricing can be found in [2] and [3].

A class of copula which better models extreme events, is time-dependent and flexible enough to take into account correlation-skew is the Marshall-Olkin copula [10]. One of the interesting features of the Marshall-Olkin copula is that it is a natural extension to the single name credit intensity model [1]. Moreover, it allows to model the default probability of each subset of a credit name set (default cluster modeling). However, to specify these probabilities for a set of \( N \) names, the Marshall-Olkin copula requires \( 2^N - 1 \) parameters. While this set up is suitable in case of few names, in case of a derivative like a CDO tranche, where the number of names is typically 125, the number of parameters grows to more than \( 10^{37} \). It is clear that, due to the exponentially increasing number of model parameters, it is very difficult to fully specify the correlation between a large number of names.

To overcome the problem of the exponentially increasing number of parameters, in [12] is presented an extension to the Marshall-Olkin copula using subordinator Lévy processes [5]. In this model, the survival probability of each name depends on Lévy subordinators via participation parameters. The definition of the Marshall-Olkin copula in terms of subordinator Lévy processes allows on the one hand, to correctly model the default clustering, and, on the other hand, to make the number of model parameters increase linearly with the number of credit names. The model financial interpretation is that each subordinator identifies a systemic risk driver, representing for example a country or an industry, and each systemic driver affects the credit state of a subgroup of the credit names. In fact, when the jump of a particular systemic factor occurs, all the survival probabilities, sensitive to the jumped factor, decrease proportionally to the participation to that factor, resulting in a positive default correlation.

A common problem in credit correlation modeling is the model calibration. The main reasons are that liquid multi-name credit instruments are very few and a typical pricing algorithm for a multi-name instrument can be quite expensive in terms of computational time, since the credit state of several names must be taken into account. The most liquid multi-name instruments are synthetic CDO tranches and each of them, according to its subordination level (equity, mezzanine, senior), reflects a different credit correlation risk. CDO tranches describe very well the credit correlation structure, therefore they can be a good calibration instrument choice. Focusing on CDO tranches as calibration instruments, a possible pricing approach is the Montecarlo simulation. However, to reach a good convergence, the Montecarlo algorithm is typically time consuming, therefore it is not suitable for a calibration routine. Other approaches have been presented in literature, see for example [4], [6], [7], [8], [14], [15], but, up to our knowledge, an analytic pricing formula in the context of subordinator Lévy Marshall-Olkin correlation has not been presented yet. For this reason, we derive in this paper two (semi)-analytic pricing formulas for synthetic CDO tranche using a Marshall-Olkin correlation model with subordinator Lévy processes.
We first note, by model independent reasoning, that the CDO tranche premium and protection leg can be expressed in terms of the integral of the CDO loss cumulative distribution function (cdf). Then, assuming a subordinator Lévy Marshall-Olkin correlation, together with some standard simplifying assumptions, we derive two (semi)-analytic expressions for the loss cdf integral. The first assumption is that all the names in the pool have the same survival probability and the same constant recovery rate. This assumption is addressed as homogeneous pool (HP) approximation. Under the HP approximation, we are able to compute the default probability of each subset of the tranche pool. The loss cdf is then given by the sum over these probabilities, and, in particular, it is a piecewise constant function, whose integration is straightforward. The resulting pricing formula can be easily implemented, does not require any numerical integration and it can be used with any class of subordinator Lévy process, the only information needed is the process Laplace transform. Finally, its computational cost does not depend on the number of subordinators chosen. Unfortunately, the numerical precision required in case of a big number of names (more than twenty) is beyond a standard machine precision, therefore this pricing formula is not suitable for large pool of names.

In order to deal with standard synthetic CDO tranches, where the number of names is typically $N = 125$, we make a further assumption. Since $N \gg 1$, we take the limit $N \to \infty$. The homogeneous pool approximation together with $N \to \infty$ is called large homogeneous pool (LHP) approximation. In this set-up, we derive a semi-analytic formula for the loss cdf integral which requires only a one-dimensional numerical integration of a well behaved function. As in case of the HP approximation, the LHP pricing formula holds for every class of subordinator Lévy process, the only information needed is the process Fourier transform. Finally, the computational cost does not increase with the number of systemic factors.

The paper is organized as follows: in section 3 we present a brief review of Lévy subordinator stochastic processes. In section 4 we describe the Marshall-Olkin copula and its generalization via subordinator Lévy processes. In section 5 we present the main results of the paper, the HP and LHP synthetic CDO pricing formulas. Finally, in section 6 we summarize the results obtained.

### 3 Lévy Processes

A Lévy process $L = \{L_t, t \geq 0\}$ is a cadlag, adapted, real valued stochastic process with $L_0 = 0$, which satisfies the following conditions (for more details on Lévy processes see [5]):

- $L$ has independent increments, i.e. for every increasing sequence of times $t_0, \ldots, t_n$, the random variables $L_{t_0}, L_{t_1} - L_{t_0}, \ldots, L_{t_n} - L_{t_{n-1}}$ are independent
- $L$ has stationary increments, i.e. for any $s < t$, $L_t - L_s$ is equal in distribution to $L_{t-s}$
- $L$ is stochastically continuous, i.e for any $\epsilon > 0$ and $t \geq 0$ it holds
  $$\lim_{h \to 0} P(|L_{t+h} - L_t| > \epsilon) = 0$$
The $L$ probability distribution is completely determined by the Fourier transform, defined as:

$\Psi(\phi, t) \equiv \mathbb{E}\left[e^{-i\phi L(t)}\right] = e^{-t\psi(\phi)}$ (1)

$\psi(\phi) = \gamma(i\phi) + \frac{1}{2}\sigma^2\phi^2 + \int_{\mathbb{R}}(1 - e^{-i\phi x} - i\phi x \mathbb{1}_{|x|<1})\nu(dx)$ (2)

where $\psi(\phi)$ is called characteristic exponent, $\gamma$ is a real-valued function, $\sigma^2 \geq 0$ and $\nu$ is a Lévy measure on $\mathbb{R}$ such that $\int_{\mathbb{R}}\min(1, x^2)\nu(dx) < \infty$. If the Lévy measure satisfies also the condition $\int_{\mathbb{R}}\min(1, x)\nu(dx) < \infty$, the process is called finite-activity, otherwise it is called infinite-activity Lévy process. Finally, as a remark, we note that the Lévy triple $(\gamma, \sigma^2, \nu)$ completely specifies the Lévy process.

3.1 Subordinator Lévy Processes

A subordinator Lévy process $S(t)$ is a non-decreasing Lévy process with a Lévy triple of the form $(\gamma, 0, \nu)$, i.e. a pure jump process (for details on subordinator Lévy processes see [5]). It is simpler to describe a subordinator Lévy process in terms of its Laplace transform rather than its Fourier transform, due to positiveness of the process:

$\left\{ \begin{array}{l}
\mathbb{E}\left[e^{-uS(t)}\right] = e^{-t[\gamma(u)+p(u)]} \\
p(u) = \int_{\mathbb{R}^+}(1 - e^{-ux} - u x \mathbb{1}_{|x|<1})\nu(dx) 
\end{array} \right.$ (3)

Usually the parameter $\gamma$ is set to satisfy the condition $\mathbb{E}\left[e^{-uS(t)}\right] = 1$, which implies $\gamma(u) = -p(u)$.

4 Subordinator Lévy Marshall-Olkin Copula

Let $\tau$ be the default time of a credit name. At time $t$, the default state of the name is represented by $\mathbb{1}_{\tau \leq t}$, which is one if the default happened before $t$, zero otherwise. The default time is the time of the first jump of a Poisson process, whose intensity is $\lambda$ (a detailed discussion on intensity credit modeling can be found in [1]). In this set-up, the survival probability of the name is

$Q(t) \equiv \mathbb{E}\left[\mathbb{1}_{\tau>\tau}\right] = e^{-\lambda t}$ (4)

and the default probability is $P(t) = 1 - Q(t)$.

In case of more than one name, the default model is given by a Marshall-Olkin copula [10], which is a natural extension to the single-name intensity model. For example, in case of two names, the corresponding intensities are given by

$\left\{ \begin{array}{l}
\lambda_1 = \tilde{\lambda}_1 + \lambda_{12} \\
\lambda_2 = \tilde{\lambda}_2 + \lambda_{12}
\end{array} \right.$
where \( \tilde{\lambda}_i, i = 1, 2 \) are the idiosyncratic intensities, and the term \( \lambda_{12} \) is the systemic intensity. The joint survival probability is

\[
Q(t_1 > t_1, t_2 > t_2) = \mathbb{E}[\mathbb{I}_{t_1 > t_1} \mathbb{I}_{t_2 > t_2}] = e^{-\lambda_1 t_1 - \lambda_2 t_2 + \lambda_{12} \min(t_1, t_2)}
\]

To properly specify the Marshall-Olkin model in case of \( N \) credit names, in addition to the idiosyncratic intensities, a systemic intensity should be given for each subset of the \( N \) names. An easy computation shows that the number of parameters needed grows exponentially with the number of names, i.e. \( 2^N - 1 \). It is clear that this approach is not suitable for large pool of names.

To overcome this difficulty, Marshall-Olkin correlation can be generalized by introducing subordinators Lévy processes [12], resulting in the so-called subordinator Lévy Marshall-Olkin correlation. In this model, the survival probability of a generic credit name \( i \) is defined as

\[
Q_i(t) = e^{-\tilde{\lambda}_i t} = e^{-\tilde{\lambda}_i t - \sum_{j=1}^{M} \phi_{ij} S_j(t)}
\]

where \( S_j(t), j = 1, \ldots, M \) are independent subordinator Lévy processes with Lévy triple \( (0, 0, \nu_j) \), identifying the systemic components, and \( \phi_{ij} \in \mathbb{R}^+ \) is an \( N \times M \) matrix representing the participation to the systemic components.

The financial interpretation of the model described by equation (5) is very simple and immediate. There are two main contributions to the survival probability of each name: an idiosyncratic factor \( \tilde{\lambda}_i \), which determines the default due to causes proper to the specific name, and a systemic factor. The systemic factor is a function of different systemic components \( S_j \), which may be thought as systemic risk drivers, like country risk, industry risk etc... Each name has its own participation to these systemic drivers via the parameters \( \phi_{ij} \). In particular, if a systemic driver increases, i.e. a subordinator jumps, the survival probability of each name will drop proportionally to the participation to that systemic factor, resulting in a positive credit correlation. The fact that only some names are sensitive to a specific risk factor generates the so-called default clustering effect.

The correlation structure already defined is completely equivalent to a Marshall-Olkin copula. Without loss of generality, we show in case of one systemic factor \( S \) and two names with participation parameters \( \phi_1, \phi_2 \), that the subordinator Lévy correlation model results in a Marshall-Olkin copula:

\[
\mathbb{E}[\mathbb{I}_{t_1 > t_1} \mathbb{I}_{t_2 > t_2}] = e^{-\tilde{\lambda}_1 t_1 - \tilde{\lambda}_2 t_2} \mathbb{E}[e^{-\phi_1 S(t_1)} - \phi_2 S(t_2)] = e^{-\lambda_1 t_1 - \lambda_2 t_2 + \eta_{12} \min(t_1, t_2)}
\]

where \( \eta_{12} = p(\phi_1) + p(\phi_2) - p(\phi_1 + \phi_2) \).

5 Collateralized Debt Obligation

A collateralized debt obligation (CDO) is a security linked to a pool of credit instruments like bonds, loans, credit default swaps, mortgage-backed securities, asset-backed securities, etc... The main purpose of a CDO is to transfer credit risk from the sponsor, the one which originates the CDO, to the investors. The originator of a CDO can be a bank,
a generic financial institution or an asset management company. To eliminate investor exposure to the sponsor credit risk, the sponsor usually creates an independent company called special purpose vehicle (SPV). The SPV purchases the debt obligations from the sponsor (in case of cash CDOs), or sells CDSs (in case of synthetic CDOs). Then it divides and packages the credit instruments, according to their credit worthiness, creating tranches. CDO tranches differ by subordination level: they are classified as equity, mezzanine, and senior. Finally, tranches are sold to investors, transferring credit risk to them. In this section we focus our attention on synthetic CDO pricing.

5.1 Synthetic CDO Tranche Cash-Flows

Let us suppose that the CDO pool is composed by \( N \) credit names. Each name has a constant recovery rate equal to \( R_i \). Then, the CDO loss function at time \( t \) is given by

\[
L_t = \frac{1}{N} \sum_{i=1}^{N} 1_{\tau_i \leq t}(1 - R_i)
\]

where \( \tau_i \) is the default time of the \( i \)-th name. A synthetic CDO tranche contract with attachment point \( \alpha_{i-1} \) and detachment point \( \alpha_i \) is composed by two legs: a protection leg \( W^{\alpha_i} \) and a premium leg \( V^{\alpha_i} \). On the one hand, whenever there is a default in the CDO pool and the loss function value is between the attachment and detachment point, the protection leg pays the loss corresponding to the defaulted name. On the other hand, the premium leg pays on a fixed schedule a fixed premium, \( \pi^{\alpha_i} \), on the remaining CDO face value.

Let us define the tranche loss \( L^{\alpha_i}_t \) as

\[
L^{\alpha_i}_t = \frac{1}{N} \sum_{i=1}^{N} 1_{L_t \in [\alpha_{i-1}, \alpha_i]} + (\alpha_i - \alpha_{i-1}) 1_{L_t > \alpha_i}
\]

then the protection and the premium leg are given by

\[
W^{\alpha_i}(t) = \mathbb{E}[\int_0^t B(u)dL^{\alpha_i}_u] = B(t)\mathbb{E}[L^{\alpha_i}_t] + \int_0^t r(u)B(u)\mathbb{E}[L^{\alpha_i}_u] \, du
\]

\[
V^{\alpha_i}(t) = \sum_{j=1}^{n_t} B(t_j)(\delta \alpha_i - \mathbb{E}[L^{\alpha_i}_{t_j}]) \Delta_j
\]

where \( B(t) \) is the discount factor, \( r(t) \) is the discount rate, assumed to be deterministic, \( \Delta_j \) is the year fraction and \( \delta \alpha_i = \alpha_i - \alpha_{i-1} \).

CDO tranches are quoted either by par-spread or by an upfront and a fixed spread. In the former case, the par-spread is given by

\[
\pi^{\alpha_i} = \frac{B(t)\mathbb{E}[L^{\alpha_i}_t] + \int_0^t r(u)B(u)\mathbb{E}[L^{\alpha_i}_u] \, du}{\sum_{j=1}^{n_t} B(t_j)(\delta \alpha_i - \mathbb{E}[L^{\alpha_i}_{t_j}]) \Delta_j}
\]

while in the latter case, given the fixed spread \( \bar{\pi}^{\alpha_i} \), the upfront \( u^{\alpha_i} \) is

\[
u^{\alpha_i} = \frac{1}{\delta \alpha_i} \left\{ B(t)\mathbb{E}[L^{\alpha_i}_t] + \int_0^t r(u)B(u)\mathbb{E}[L^{\alpha_i}_u] \, du - \bar{\pi}^{\alpha_i} \sum_{j=1}^{n_t} B(t_j)(\delta \alpha_i - \mathbb{E}[L^{\alpha_i}_{t_j}]) \Delta_j \right\}
\]
From the expressions of the protection leg (7) and the premium leg (8), we notice that the only non-trivial quantity to compute is the expected value of the tranche loss $\mathbb{E}[L^\alpha_t]$, which, after a little algebra, can be expressed in terms of the loss cumulative distribution function $F_{L_t}(x)$:

$$\mathbb{E}[L^\alpha_t] = \int_{\alpha_{i-1}}^{\alpha_i} (x - \alpha_{i-1}) dF_{L_t}(x) + \delta \alpha_i (1 - F_{L_t}(\alpha_i)) = \delta \alpha_i - \int_{\alpha_{i-1}}^{\alpha_i} F_{L_t}(x) dx$$

For an easier notation, we define

$$\Sigma_{L_t}(a, b) \equiv \int_a^b F_{L_t}(x) dx$$

Finally, substituting (9) into (7) and (8), and using definition (10), the premium and the protection leg read

$$W^\alpha(t) = \delta \alpha_i - \Sigma_{L_t}(\alpha_{i-1}, \alpha_i) B(t) + \int_0^t \Sigma_{L_u}(\alpha_{i-1}, \alpha_i) dB(u)$$

$$V^\alpha(t) = \sum_{j=1}^{n_t} B(t_j) \Sigma_{L_{t_j}}(\alpha_{i-1}, \alpha_i) \Delta_j$$

With some simple algebra and model-independent reasoning, the expressions of the premium leg and protection leg are now written as function of the loss cdf integral. At this point, in order to find $F_{L_t}(x)$, a stochastic model for the loss function is needed.

### 5.2 Loss Cumulative Distribution Function

In the following we assume a subordinator Lévy Marshall-Olkin correlation model, introduced in section 4, to define the correlation between credit names. In particular, recalling (5), the survival probabilities are given by

$$Q_i(t) = \mathbb{E}[\mathbf{1}_{\tau_i > t}] = e^{-\tilde{\lambda}_i t - \sum_{j=1}^{M} \phi_{ij} S_j(t)}, \quad i = 1, \ldots, N$$

where $\tilde{\lambda}_i$ are the idiosyncratic intensities, $\phi_{ij}$ are the participation factors and $S_j(t)$ are subordinator Lévy processes with Lévy triple $(0, 0, \nu_j)$, independent from each others.

To compute the loss cdf we proceed in two steps: the first one is to condition the loss function to the realization of the systemic factors $S_j(t)$

$$L_t|s_1, \ldots, s_M = \frac{1}{N} \sum_{i=1}^{N} (1 - R_i) \mathbf{1}_{\tau_i \leq t}|s_1, \ldots, s_M$$

and compute the conditional loss cdf $F_{L_t}(x|s_1, \ldots, s_M)$. The second step is then to integrate the conditional loss cdf against the subordinator probability distribution functions

$$F_{L_t}(x) = \int_{\mathbb{R}^M} d^M s F_{L_t}(x|s_1, \ldots, s_M) \prod_{k=1}^{M} \Phi_{S_k}(s_k)$$

where $\Phi_{S_k}$ is the probability distribution of the $S_k$ subordinator.
5.2.1 Homogeneous Pool Approximation

In order to simplify the computation we assume that all the names in the CDO pool have the same constant recovery rate \( R_i = R \) and the same survival probability

\[
Q_i(t) = e^{-\tilde{\lambda}t - \sum_{j=1}^{M} \phi_j S_j(t)}, \quad i = 1, \ldots, N
\]

(15)

This approximation is addressed as homogeneous pool (HP) approximation. Conditioned to the realization of the subordinators, the defaults are independent events. Therefore, the conditional loss is distributed according to a binomial random variable, where the single name default probability \( P_t(s) \) is given by

\[
P_t(s) = 1 - e^{-\tilde{\lambda}t - \sum_{j=1}^{M} \phi_j s_j}
\]

(16)

The conditional probability that \( j \) names in the pool default is

\[
P(j|s) = \binom{N}{j} [P_t(s)]^j [1 - P_t(s)]^{N-j}
\]

from which, after some calculation, we can write the conditional loss cdf as

\[
F_{L_t}(x|s_1, \ldots, s_M) = \sum_{j=0}^{n_x} \sum_{k=0}^{j} (-1)^{j-k} \binom{N}{j} \binom{j}{k} e^{-(N-k)\tilde{\lambda}t} e^{-(N-k)\sum_{l=1}^{M} \phi_l s_l}
\]

(17)

where \( n_x = \frac{xN}{(1-R)} \). Substituting (17) into (14), and after a little algebra, we have

\[
F_{L_t}(x) = \sum_{k=0}^{n_x} \beta_{n_x,k} B_k(t)
\]

(18)

where

\[
B_k(t) = \binom{N}{k} e^{-(N-k)\tilde{\lambda}t - \sum_{j=1}^{M} p_l (N-k)\phi_j}, \quad \beta_{n,k} = \sum_{j=0}^{n-k} (-1)^{j} \binom{N-k}{j}
\]

and \( p_l(x) \) is the Laplace exponent of the \( l \)th subordinator, defined in (3). We note that the loss cdf (18) is a piece-wise constant function, therefore its integral (10) can be easily evaluated.

The homogeneous pool approximation gives an analytic formula for the loss cdf integral, which, in particular, is the integral of a piece-wise constant function, whose computation is straightforward. Moreover, equation (18) holds for every class of subordinator Lévy process, the only information needed is the process Laplace transform. Finally, the computational cost does not depend on the number of systemic factors.

As a drawback, we notice that coefficients \( \beta_{n,k} \) present a sum of, potentially, very huge numbers with alternating sign. Unfortunately, in case of a big number of names, the result of this sum cannot be correctly evaluated by a calculator with a standard machine precision due to numerical errors. Therefore, this formula can be applied only in case of small pool of names, typically less than twenty. To be able to deal with larger pools, another approximation is needed and it will be shown in the next section.
5.2.2 Large Homogeneous Pool Approximation

As already said, the HP formula (18) can be numerically evaluated with a good precision only in case of small pool of names. However, a standard synthetic CDO has a pool composed by $N = 125$ names. In order to correctly price also CDO tranches with pool of names much bigger than the limit allowed by the HP formula, in addition to the homogeneous pool assumption, we make a further approximation: since $N \gg 1$ for standard synthetic CDOs, we take the limit $N \to \infty$. In this limit, the loss function (13) becomes an average of infinite i.i.d. random variables, because the default times conditioned to the subordinator realizations are i.i.d. According to the central limit theorem, the conditional loss function tends to its mean with probability one:

$$\lim_{N \to \infty} F_{L_t}(x|s_1, \ldots, s_M) = \theta \left( x - (1 - R)(1 - e^{-\tilde{\lambda}t - \sum_{k=1}^{M} \phi_k s_k}) \right)$$

where $\theta(x)$ is the Heaviside step function. The assumption that the survival probability is the same for each name in the pool, together with the approximation $N \to \infty$ is called large homogeneous pool (LHP) approximation. Substituting equation (19) in equation (14) we have

$$F_{L_t}(x) = \int_{\mathbb{R}^M} d^M s \theta \left( x - (1 - R)(1 - e^{-\tilde{\lambda}t - \sum_{k=1}^{M} \phi_k s_k}) \right) \prod_{k=1}^{M} \Phi_{S_k}(s_k)$$

It is better to write integral (20) in terms of the subordinator Fourier transform (1):

$$F_{L_t}(x) = \int_{\mathbb{R}^M} d^M u \prod_{k=1}^{M} \Psi_{S_k}(u_k) \int_{\mathbb{R}^M} d^M s \theta \left( x - (1 - R)(1 - e^{-\tilde{\lambda}t - \sum_{k=1}^{M} \phi_k s_k}) \right) e^{\sum_{k=1}^{M} u_k s_k}$$

After a simple computation, we get

$$F_{L_t}(x) = \frac{1}{2} - \frac{i}{2\pi} \int_{\mathbb{R}} \frac{du}{u} \left( 1 - \frac{x}{1 - R} \right)^{-iu} e^{-iu\tilde{\lambda}t} e^{-t\sum_{k=1}^{M} \varphi_k(\phi k u)}$$

where $\varphi_k(x)$ is the characteristic exponent, defined in (2), of the $k-th$ subordinator. We are now able to compute the integral of the loss cdf (10):

$$\Sigma_{L_t}(a, b) = \frac{b - a}{2} + \frac{i(1 - R)}{2\pi} \int_{\mathbb{R}} \frac{du}{u(1 - iu)} e^{-iu\tilde{\lambda}t} e^{-t\sum_{k=1}^{M} \varphi_k(\phi k u)} \cdot \left\{ \left( 1 - \frac{b}{1 - R} \right)^{1-iu} - \left( 1 - \frac{a}{1 - R} \right)^{1-iu} \right\}$$

After some re-arrangement, we can write (22) as

$$\Sigma_{L_t}(a, b) = \frac{b - a}{2} - \frac{1 - R}{\pi} \int_{0}^{\infty} du \ g_t(u, a, b)$$

$$g_t(u, a, b) = Im \left( \frac{e^{-iu\tilde{\lambda}t - \sum_{k=1}^{M} \varphi_k(\phi k u) t}}{u(1 - iu)} \left\{ \left( 1 - \frac{b}{1 - R} \right)^{1-iu} - \left( 1 - \frac{a}{1 - R} \right)^{1-iu} \right\} \right)$$
Finally, substituting (23) in (7) and (8), we get expressions for the CDO protection and premium leg

\[
W^{\alpha_i}(t) = \frac{\delta \alpha_i}{2} - \frac{1 - R}{\pi} \int_0^t dB(s) \int_0^\infty du g_s(u, \alpha_{i-1}, \alpha_i) \tag{25}
\]

\[
V^{\alpha_i}(t) = \sum_{j=1}^{n_t} B(t_j) \Delta_j \left[ \frac{\delta \alpha_i}{2} - \frac{1 - R}{\pi} \int_0^\infty du g_{t_j}(u, \alpha_{i-1}, \alpha_i) \right] \tag{26}
\]

Some final remarks on equation (23): we first note that the computational cost does not depend on the number of systemic factors, in particular the computational effort always requires only a one-dimensional numerical integration. Moreover, even if the integrand function (24) is an oscillatory function, it admits a finite limit in \(u = 0\), and it goes rapidly to zero as \(u \to \infty\). These properties allow an accurate integration using any of the most common numerical integration routines. Finally, as in case of the HP approximation, equation (23) holds for each subordinator class, the only information needed is the process Fourier transform.

6 Conclusions

The main results of this work are two (semi)-analytic synthetic CDO tranche pricing formulas in the context of the subordinator Lévy Marshall-Olkin correlation model. The two formulas are particularly suitable for the correlation model calibration because their evaluation is very fast in terms of computational time.

To derive these formulas, we first express the CDO tranche premium and protection leg in terms of the loss cdf integral. Then, under some simplifying assumptions, we are able to analytically compute the loss cdf integral. The first formula is derived under the homogeneous pool approximation, where all the names have the same recovery rate and survival probability. In this set-up, the default probability of each subset of the tranche pool can be analytically computed. The loss cdf is then the sum of these probabilities, which results in a piece-wise constant function, whose integration is straightforward. The main advantages of this expression is that it does not require any numerical integration and it holds for every class of subordinator Lévy process, the only information needed is the process Laplace transform. Finally, the computational cost does not depend on the number of subordinators. Unfortunately, in case of large pools, this formula cannot be correctly evaluated because the numerical precision required is beyond a standard machine precision.

To deal also with pools composed by many credit names, we derive a second pricing formula adopting the large homogeneous pool approximation. Under this approximation, we have a semi-analytic formula for the loss cdf integral, whose computation requires only a one-dimensional numerical integration, independently of the number of subordinators. The integrand function is well behaved and the integral can be easily computed with any numerical integration routine. Finally, as in the homogeneous pool case, the loss cdf integral expression holds for every subordinator Lévy process, the only information needed is the process Fourier transform.
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