A risk-neutral approach for the evaluation of NPLs

Danilo Tiloca
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A. ANNEX 33
A risk-neutral approach for the evaluation of NPLs

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Group Special Entitles Risk and Evaluation

Abstract

This paper describes a new methodology which allows banks to evaluate non-performing loans (NPLs) using a risk-neutral approach. In more detail, it illustrates the methodological framework behind the definition of the risk-neutral expected loss used to estimate the loans fair value. The risk-neutral expected loss is calculated by extending the Merton framework for modeling corporate liabilities. The proposed risk-neutral approach is suitable at producing estimates, in a fair value computation context, that are as close as possible to the exit price as mandated by IFRS-13 and is not biased by the stressed market conditions often observed on defaulted debt. In this context the pricing model is able to provide good estimates especially when applied to large portfolios (where the idiosyncratic risk is low) and is requiring few parameters to work: the expected recovery, the recovery time, the market risk-premium and the default vintage. A model that can help the banks to estimate a “fair” price for this kind of asset can help in reducing the difference between the asked price (for a bank often is the book value) and the offered market price, facilitating the development of a market for NPLs.

Keywords:
NPL, non-performing loans, risk premia, Merton model, fair value, credit risk.

JEL Codes:
G12, G13
1. Introduction

The introduction of IFRS-9, at the beginning of 2018, requires that loans that will fail the SPPI test\(^1\), are evaluated at fair value. The test also has to be applied to restructured non-performing loans (NPLs) and this implies the need to develop a model able to provide risk-neutral prices also for the NPLs. The new standard is introducing in this way the possibility of an evaluation at fair value for the loans that was not required in the previous regulations. But while for the evaluation of performing loans are available well establish methodologies\(^2\), for the evaluation of the non-performing loans (NPLs) it is missing a well consolidated evaluation framework suitable to be used for the evaluation at fair value of this kind of assets. The need for the development of a market model for NPLs has become even more compelling as a consequence of the last financial crisis that left most banks with a significant increase of non-performing exposures. The recent experience of the non-performing loan crises (for example Ireland and Spain) shows that the disposal of these loans appears to be an effective means of cleaning up bank balance sheets. In Italy one of the main reasons for the failure to develop a secondary market for non-performing loans is due to a substantial difference between the book value of these assets and the prices offered by investors. Sometimes the difference is so great that it is difficult for a bank to understand if the offered price can be considered “fair” under the standard market conditions. So a model that can help the bank to estimate a “fair” price for this kind of assets can help in reducing the difference between the asked price (for a bank often this is the book value) and the offered market price. Theoretically it could be possible to measure the market value of distressed debt shortly after default (see Altman and Kishore (1996)) in order to estimate the market price of the recovery, but with the only exception for relatively few large corporates, most of the defaulted bank’s customers do not have debt that is traded publicly. This left the banks with a need to estimate the post-default cash flows and to discount them to an appropriate rate to calculate the market price. However, the discount rate to be applied to post-default cash flows is a largely unresolved issue, among both practitioners and market participants.

The problem of evaluating the appropriate discount rate for the workout recoveries has already been addressed by many researchers in literature, especially for regulatory purposes of estimating the economic LGD. But there are few papers addressing the problem of evaluating the pricing of NPLs\(^3\). Among the proposed methods of estimating the economic LGD, those based on examining the ex-post returns on defaulted debt appears to be the most appropriate for developing a pricing model for NPLs. Unfortunately, even in this case they are missing a general agreement on the most appropriate discount rate. In Araten et al. (2004) is proposed a fixed 15% discount yield that is supported by the

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\(^1\) SPPI is the acronym of “Solely Payments of Principal and Interest” and the test is proposed by the new IFRS-9 standard to verify if contractual terms of a financial asset (as a whole) give rise to cash flows that are solely payments of principal and interest on the principal amounts outstanding i.e. cash flows that are consistent with a basic lending arrangement.

\(^2\) See the paper of Skoglund (2017) for an interesting overview, with an extended bibliography, of the methodologies that can be used to evaluate the fair value of performing loans.

\(^3\) As an example of papers proposing an evaluation model for the NPLs in Italy using a DCF model see: Ciavoliello et al. (2016) and Humblot (2017).
observation of the average return demanded by the specialized investors on distressed debt, while in Brady et al. (2007) is proposed for the bank debt, a spread of 9.4% above the risk free rate. Finally Jacobs (2012) using Moody’s Ultimate Recovery Database (1987–2010) estimates an average “realized” yield$^4$ for the bank loans of 32.6% and of 28% for the total of loans and bonds analyzed. But it is not clear how the observation of these “realized” returns can be translated in a “fair” price for selling NPLs. The dispersion of the returns observed by Jacobs range from -100% to 894% and it is difficult to think that there are buyers willing to accept a return of -100% or sellers willing to accept a return of 894%. The average values could help in defining an average market yield, but even in this case there are problems when the analysis is performed in more detail. For example Jacobs reported that the “realized” yield is significantly higher for secured loans as compared to unsecured loans, 33% versus 20% respectively$^5$. This is the opposite of what is observed in NPLs transactions where the quality of collateral is properly taken into consideration and the yield asked on secured loans is lower than the one asked on unsecured loans. The problem is that the discount approach is of limited applicability when is applied on an illiquid market given the impossibility for these models to provide a “structural” explanation of the discount margin or internal rate of return. In the literature there are reports of several attempts to use the “structural” approach to explain the observed yields on defaulted debt (see Jacobs (2009) for an extensive review), but it seems that the model based approaches that invoke a CAPM structure, whether “structural” calibration or regression, generate discount rate estimates significantly lower than purely empirical approaches (see Jacobs (2009), page 30) and so it seems that these models are not suitable to be used in this context.

In this paper we don’t focus on the complexity behind the estimation of the workout recoveries and it is assumed that this estimation is performed taking properly into account the quality and seniority of collateral, the business cycle and the business sector to which the borrower belongs to. The workout recovery estimated using historical data and the previous assumptions is often referred as estimated under the physical measure or real-world measure$^6$ and the main purpose of the paper is the development of a formula that allows to easily transform the recovery estimated under the physical measure in a risk-neutral recovery estimate.

To develop a formula suitable for this transformation, will be used the Merton’s “structural” approach, to model the recovery process post-default. Using this approach, it is shown that there is a link between the real-world expected loss and the risk-neutral one (observed and embedded in the market spreads) and that this link is a function of the sharpe-ratio. The proposed methodology estimates, through the calibration, the sharpe-ratio using quotes observed in the loans primary market. The main assumption of this approach is that the investors on NPLs will require at least the same risk-

$^4$ In the paper is called RDD (Return on defaulted debt).

$^5$ This difference could be due to the fact that, as reported by Schuermann (2004), the bond market prices are pricing in the APR (absolute priority rule) violations, which allows the owner of unsecured debt to receive more that should be expected if APR is respected. The violation of APR implies a lower recovery for the secured debt and this violation could justify the difference on “realized” yields for secured and unsecured debt. However this violation could be relevant only for the large corporates while the SME and retail loans having a simpler capital structure should not be affected by this problem.

$^6$ In this paper the terms real-world expected loss (recovery) and physical expected loss (recovery) are interchangeable.
premium implied by the sharpe-ratio observed in the loans primary market and this approach will solve the problem related to the calibration of “structural” models highlighted in Jacobs (2009). The estimated fair value calculated in this way, has the aim of determining the exit-price for the non-performing loans and is not biased by the “stressed” prices observed on the defaulted debt. The market price of risk estimated in this way reflects all the components that a market participant would consider when pricing the NPL: expected loss, the total recovery time, the uncertainty of EL, the default vintage and the funding cost.

The paper is structured as follows. After the introduction of the main concepts underlying the risk-neutral approach, the document continues with a more technical section on the theoretical framework, here are provided all the demonstrations of the main formulas used to pricing the loans and the NPLs, in particular this section contains an overview of Merton’s model and its main results. Then, to be able to explain the risk-premium asked by the investors on NPLs, it is proposed to model the recovery process after default as a stochastic process, in particular this section contains a detailed proof of the equation used to transform the real-world expected loss in a risk-neutral expected loss. This assumption will allow to define a bridge function that will enable the pricing of the NPLs in the risk-neutral framework. Finally, are provided some empirical results of usage of the “structural” model proposed in this paper. A summary of the methodology is provided at the end of the document.
2. Valuation Framework

2.1 NPLs Accounting Rules and Fair Value calculation

All the main European banks that adopt the international accounting standards calculate the net book value of non-performing loans using the amortized cost method where the expected recovery flows $\hat{f}(t)$ are discounted using the original effective interest rate $r_e$. Therefore, the NPL net book value (NBV) is given by the following expression:

$$NBV = \sum_{t=1}^{T} \hat{f}(t) e^{-r_e t}$$  \hspace{1cm} (2.1)

When a debtor defaults, the bank must assess the recoverable amount $f(t)$, and the recovery time $t$. Usually the recoverable amount is calculated net of the workout expenses, while the contractual interest flows due after the default are not included since are legally lost (interest income foregone). In the remaining part of the document, the convention adopted will be that $R_T$ represents a fixed fraction of the notional outstanding $N$ at the moment of default and so $f(t)$ represents a given fraction of outstanding debt $N$, expected to be recovered at time $t$. Under this convention, the expected recovery rate is defined as:

$$R_T = \sum_{t=1}^{T} f(t), \quad \text{with} \quad f(t) = \frac{\hat{f}(t)}{N}$$  \hspace{1cm} (2.2)

accordingly, the bank’s estimated fractional expected loss $EL_T$ is given by the following expression:

$$EL_T = 1 - R_T$$  \hspace{1cm} (2.3)

When it is required to evaluate the NPLs at fair value in the equation (2.1) it is necessary to substitute the original effective interest rate $r_e$ with an appropriate market yield $y$ that has to compensate the investors for the time value of money and for the risks related to the uncertainty of the estimation and timing of expected recoveries. Therefore, the fair value (FV) of a NPLs is given by the following expression:

$$FV = N \cdot \sum_{t=1}^{T} f(t) e^{-y t}$$  \hspace{1cm} (2.4)
Since for non-performing loans, the estimates regarding the timing and size of collections have greater uncertainty than the estimate of total recovery, it is better to define the fair value as a function of $R_T$. To do that it is possible to use the Taylor’s expansion of the exponential function to approximate the above equation as

$$FV = N \left( \sum_{t=1}^{T} f(t) - y \sum_{t=1}^{T} f(t) t \right) = N \cdot R_T \left( 1 - y \sum_{t=1}^{T} \frac{f(t) t}{R_T} \right)$$

by inverting the Taylor’s expansion, this expression can be rewritten as

$$FV = N \cdot R_T \cdot e^{-y t_m}$$  \hfill (2.5)

with

$$t_m = \frac{\sum_{t=1}^{T} f(t) t}{R_T}$$  \hfill (2.6)

it is possible to write the above expression as a function of the market spread over the risk-free rate by defining the yield as:

$$y = r(t_m) + s$$  \hfill (2.7)

where $r(t)$ represents the risk-free rate and $s$ the market spread implied by the market’s transaction prices. With this new definition the fair value can be calculated as

$$FV = N \cdot RR_T \cdot e^{-r(t_m) t_m}$$  \hfill (2.8)

where $RR_T$ represent the risk-neutral recovery and is defined as

$$RR_T = R_T \cdot e^{-r t_m}$$  \hfill (2.9)

This equation satisfies the requirement that when the physical cash flows are adjusted for the investor’s risk aversion so that the recoveries are calculated under risk-neutral measure, the default-free term structure is the most appropriate for discounting.

The calculation of the fair value of non-performing loans is made difficult due to the lack of information related to the few transactions observed on the NPL market. Being the NPLs non-standard products, without knowing: the default vintage, the expected recovery and time to recovery; the only
knowledge of the price is not enough for a reliable estimate of the market’s spread. In such a situation, it is necessary to develop a theoretical model that is able to calculate the market’s risk-premium using variables that can be observed with a good degree of reliability and that are not biased by the “stressed” market condition often observed on defaulted debt. The proposed risk-neutral approach has been developed to solve the problem of providing good fair value estimates even when the reference market lacks of observability. Since the proposed model represents an extension of the Merton framework applied to the calculation of the risk-neutral default probabilities, in the next section will be described how the Merton framework works and how this can be extended to the NPLs evaluation.

The IFRS-13 requires that when a price for an identical asset is not observable, an entity measures fair value using another valuation technique that maximizes the use of relevant observable inputs. Because fair value is a market-based measurement, it is measured using the assumptions that market participants would use when pricing the asset, including assumptions about risk (IFRS-13-3). In this paper will be shown, that a good proxy for the risk-premium to be applied for the NPLs evaluation, can be estimated from the loans primary market.
3. Background on risk-neutral pricing for performing loans

To mark credit assets using a valuation model it is necessary to estimate the risk-neutral default probabilities from the quotes observable in the credit market (bond prices, credit default swap spreads). Currently, two approaches to mark a credit asset to model are proposed in the literature: 1) the reduced-form approach, where it is assumed that the price of any security is the expected value of its future cash flows, and that is compatible with the discount cash-flows (DCF) methodology 2) the structural approach, where the equations come from an underlying economic (or physical) model. Structural estimation is an estimation which uses these equations to identify parameters of interest. The Merton model is considered a “structural” model because it is assumed that the default occurs when the value of a firm’s assets (structural parameter) declines below a given threshold (economic condition). The first approach is applicable on liquid credit markets where it is possible to observe the implied discount spread and there is no need to provide a theoretical explanation of the observed spreads. The second approach is the most appropriate when it is necessary to evaluate those loans whose prices are not observable in the market, since is able to provide a “structural” explanation of the implied discount spread. In this section there will be provided an overview of structural Merton’s model, that will be subsequently extended in the next section to price the NPLs.

3.1 Merton’s Risk Neutral Default Probability

In the Merton model (see Merton [1974]) , it is assumed that a company asset value (identified by the subscript $i$) can be described by the following Brownian motion:

$$\frac{dA_i}{A_i} = \mu_i \cdot dt + \sigma_i \cdot dW$$

(3.1)

if all the assumptions necessary to apply the Black-Scholes framework are satisfied it is possible to find the following solution for the stochastic process $A(t)$ over the period $[0,t]$:

$$\ln \left( \frac{A_t}{A_0} \right) = \left( \mu_i - \frac{1}{2} \sigma_i^2 \right) t + \sigma_i \sqrt{t} \cdot \epsilon, \quad \epsilon \sim N(0,1)$$

(3.2)

Now, to find the risk-neutral default probability, it is necessary link the value of asset $A_i$ to a default condition. The Merton’s model assumes that if the value of asset $A_i$ decreases below a given
threshold $B_i$ (that represents the outstanding debt) there is the default. It is well known (see for example Skoglund and Chen (2015), pag. 182-183) that in the Black-Scholes framework this probability can be calculated as:

$$\text{Prob}(A_{i,T} < B_i) = \Phi(\varepsilon)$$

where

$$\varepsilon = \frac{\ln(B) - \left( \ln(A_{i,0}) + \left( \mu_A - \frac{1}{2} \sigma_A^2 \right) T \right)}{\sigma_A \sqrt{T}}$$

The equation (3.2) describes the process in the real-world and hence it depends on the real world drift $\mu_A$. It is possible to use the Gisarnov’s Theorem to change the drift $\mu_A$ in the risk-free rate $r$, so that it is possible define the risk-neutral cumulative probability $CDP_{T,Q}$ as:

$$CDP_{i,T}^{Q} = \Phi \left( \varepsilon + \frac{\mu_A - r}{\sigma_A} \sqrt{T} \right)$$

(3.3)

by using the eq.(3.3) it is possible to define the relationship suitable to transform the real-world default probabilities in risk-neutral default probabilities as:

$$CDP_{i,T}^{Q} = \Phi \left( \Phi^{-1}(CDP_{i,T}^{D}) + \mu_A - r \right)$$

(3.4)

The term $(\mu_A - r) / \sigma_A$ is the market risk-premium or sharpe-ratio related to the specific asset $A_i$ and in the rest of the paper this will be indicated by the symbol lambda ($\lambda_i$). This parameter represents the market risk-premium over the risk free rate $(r)$ per unit of risk:

$$\lambda_i = \frac{\mu_A - r}{\sigma_A}$$

(3.5)

By using the definition (3.5) it is possible to write the expression (3.4) as:

$$CDP_{i,T}^{Q} = \Phi \left[ \Phi^{-1}(CDP_{i,T}^{D}) + \lambda_i \sqrt{T} \right]$$

(3.6)

In this paper the estimation of the sharpe-ratio will use the prices observed on the primary loan market and therefore the equity market will not be taken into consideration. In the rest of the paper the term lambda and (credit) market risk-premium (per unit of risk) will be often used as synonyms.
This is the monotone function, derived in the Merton framework, which will allow to transform the *real-world* default probabilities in *risk-neutral* default probabilities. This expression can be used, as proposed by Demchak (2000), Crouhy et al. (2001) and Vasicek (2002), to price loans using a *risk-neutral* approach.  

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It is noteworthy that a recent analysis performed by Berg (2009) has shown that the monotone function (3.6), derived in the Merton framework, to transform real world PDs and recovery rates into credit spreads are also a very good approximation in more advanced structural models of default and this good approximation may be an explanation for the good pricing performance of these models.
4. Risk-neutral pricing for non-performing loans

4.1 The Post-Default Recovery Process

In this section is proposed a "structural" approach for estimating the expected loss in a risk-neutral framework. A similar approach was also proposed by Maclachlan (2004), Pykhtin (2003), Barco (2007) and more recently by Jacobs (2009). In the proposed approach will not be modeled the portfolio correlation since the purpose of this article is to develop a pricing function to evaluate each non-performing loan, and it is market practice to evaluate a portfolio of NPLs as the sum of the value of each loan\footnote{The same practice is, for example, in place when it is necessary to evaluate a portfolio of CDS or a portfolio of equity options.}. The application of the Black-Scholes framework in this setting will allow to derive a closed formula for the recovery risk premium.

Since after a default the most important variable of estimation is the loss given default (LGD), and as a consequence, its estimation is subject to the uncertainty of the recovery process. It is assumed that the fractional recovery can be described by a stochastic process $R_{i,t}$ in a Black-Scholes framework. In effect the fractional Recovery $R_{i,t}$ can be expressed as the ratio of the value of assets $A_{i,T}$ at the end of the recovery period $T$ and the outstanding debt $B_i$, that at the moment of default is a known value\footnote{In this section and in the subsequent sections the subscript $i$ will be used both for a single loan and for a cluster of loans that are homogeneous with respect to the recovery risk.}. Thus this model can be considered as an extension of the structural Merton framework where the value of the recoverable asset is described by the following stochastic process\footnote{The application of the Merton model on performing loans has been criticized because it does not allow early defaults, for the non-performing loans this limitation does not apply.}:

$$\frac{dR_{i,t}}{R_{i,t}} = \mu_{R_i} dt + \sigma_{R_i} dW$$

(4.1)

with:

$$R_{i,t} = \frac{A_{i,t}}{B_i}$$

even if $A_{i,t}$ could be described by the stochastic equation (3.1), after the default it is assumed that the drift $\mu_{R_i}$ and the volatility $\sigma_{R_i}$ could be different and for this reason are identified with different symbols\footnote{In effect the event of foreclosure can impart a shock to the asset value. In some cases, the bankruptcy can help the firm to reduce the pension liability or certain contracts could be invalidated. Recently Carey and Gordy (2016) proposed a model to incorporate foreclosure shocks to the recovery value.}. Since it was observed that usually when the NPLs are sold on the market, the investors are asking a huge market risk-premium. It is interesting to study in this framework, the difference between...
the expected recovery estimated by the bank \( (R_{i,T}) \) and the expected recovery estimated by the market in a risk-neutral framework \( (R_{i,T}) \) at the end of the recovery period \( T \). The difference between \( R_{i,T} \) and \( R_{i,T} \) can be interpreted as the recovery risk-premium \( P_{i,0} \) that the investor asks as a compensation for the recovery risk embedded in the estimation of ultimate recovery\(^1\). In mathematical terms it is possible calculate this difference as:

\[
P_{i,0} = R_{i,T} - E^Q \left[ R_{i,T} \right] - E^Q \left[ R_{i,T} \right] \]

(4.2)

In the above expression, it has been assumed reasonable, that the investor seeks protection only if the actual recovery observed at the end of the recovery period is lower than that estimated by the bank. This assumption is justified by the fact that when a bank sells a large stock of private loans, investors perform a due diligence on the portfolio for sale, at the end of this activity, the real-world estimate of \( R_{i,T} \) should be the same as the estimate made by the bank. It is possible calculate \( P_{i,0} \) as a put-option with strike equal to the bank’s value of the expected recovery \( R_{i,T} \). In effect since the expected future recoveries are not observable, it can be assumed that if pricing is rational then such realized cash flows should on average coincide with their expectations\(^1\). In a risk-neutral measure the drift \( \mu_{R_i} \) is substitute by the risk-free rate \( r \) so that:

\[
R_{i,T} = E^Q \left[ R_{i,T} \right] = R_{i,0} e^{rT}
\]

(4.3)

it is well known (see for example Hull (2002), pag. 246-247) that in a Black-Scholes framework the value of this put-option is given by the following equation:

\[
P_{i,0} = e^{-rT} \cdot R_{i,T} \cdot \Phi \left( d + \sigma_{R_i} \sqrt{T} \right) - R_{i,0} \cdot \Phi \left( d \right)
\]

(4.4)

where

\[
d = \frac{\ln \left( \frac{R_{i,T}}{R_{i,0}} \right) - \left( r + \frac{1}{2} \sigma_{R_i}^2 \right) T}{\sigma_{R_i} \sqrt{T}}
\]

(4.5)

\(^1\)The term ultimate recovery has been introduced in Emery et al. (2007) and in this paper it is equivalent to the total recovery as defined in (2.2) “realized” at the end of the workout process.

\(^1\)On the single loan, significant differences could be observed, but these differences disappear when the average is calculated at portfolio level. Furthermore, investors often only provide risk-neutral estimates. In this case, it is possible to reverse the fair value formula proposed in this document, to get the implied real-world recovery estimated by investors.
as expected, the loss risk-premium $P_{i,0}$ that the investors are asking, is strictly linked to the uncertainty of the estimation of $R_{i,t}(T)$. This is even more clear if in (4.5) is substitute the value of (4.3), so that the value of $d$ simplifies in:

$$d = -\frac{1}{2}\sigma_k \sqrt{T}$$

and $P_{i,0}$ can be written as:

$$P_{i,0} = R_{i,0} \left\{ \Phi \left( \frac{1}{2}\sigma_k \sqrt{T} \right) - \Phi \left( -\frac{1}{2}\sigma_k \sqrt{T} \right) \right\}$$

(4.6)

In the above expressions $R_{i,0} = R_{i,T} e^{-r_i T}$ represent the present value of estimated recovery $R_{i,T}$ at the end of the recovery period $T$ in a risk-neutral world. Usually $R_{i,0}$ is already taking into account the recovery costs. In case it is necessary to consider additional servicing recovery costs, it is possible to consider them by replacing in the above equations $R_{i,0}$ with $R_{i,0} e^{-q_i T}$, where in this case $q_i$ represents the continuous compounding annualized servicing recovery costs.

Now given the fact that both $R_{i,T}$ and $R R_{i,T}$ are undiscounted values that represent the recovery at the end of the workout process, it is necessary to consider the undiscounted value $P_{i,T}$ to define the relationship between them

$$RR_{i,T} = R_{i,T} - P_{i,T}$$

(4.7)

with

$$P_{i,T} = R_{i,T} \left\{ \Phi \left( \frac{1}{2}\sigma_k \sqrt{T} \right) - \Phi \left( -\frac{1}{2}\sigma_k \sqrt{T} \right) \right\}$$

(4.8)

in an equivalent manner the risk-neutral expected loss at the end of the workout period is given by the following equation:

$$ELR_{i,T} = EL_{i,T} + P_{i,T}$$

so that it is possible to write the following expression for the risk-neutral expected loss:
\[ ELR_{i,T} = EL_{i,T} + \left(1 - EL_{i,T}\right) \left\{ \Phi \left( \frac{1}{2} \sigma_R \sqrt{T} \right) - \Phi \left( -\frac{1}{2} \sigma_R \sqrt{T} \right) \right\} \] \hspace{1cm} (4.9)

and for the risk-neutral recovery:

\[ RR_{i,T} = R_{i,T} - R_{i,T} \left\{ \Phi \left( \frac{1}{2} \sigma_R \sqrt{T} \right) - \Phi \left( -\frac{1}{2} \sigma_R \sqrt{T} \right) \right\} \] \hspace{1cm} (4.10)

It can be concluded that, after the default, the risk-neutral expected loss \( ELR_{i,T} \) is always greater than the bank’s expected loss \( EL_{i,T} \) and the difference is a growing function of volatility of \( R_{i,t} \) and the total recovery time \( T \).

The post-default risk-neutral expected loss \( ELR_{i,T} \) calculated in this section requires the knowledge of the volatility \( \sigma_{R_i} \). To be able to provide risk-neutral estimates for the recovery risk-premium the volatility \( \sigma_{R_i} \) should be estimated from market prices, but most of the defaulted bank’s customers do not have debt that is traded publicly. For this reason, in the next section the proposal of a bridge function is given, that allows a connection in the equation (4.9), describing \( ELR_{i,T} \) in the post-default state, with an analogous equation evaluated in the pre-default state. The bridge function will allow to move from one state to the other with continuity and has the significant advantage that the market’s risk-premium observed in the pre-default world can be migrated to the post-default with a good level of accuracy.

4.2 Extending the Merton framework to NPLs

In this section is proposed an extension of the Merton equation (3.6) to the NPLs. The main assumption behind this extension is that when a loan default the uncertainty on the estimation of real-world default probability is substituted by the uncertainty on the expected loss, as shown in the previous section. In effect by using the transactions observable on the primary loans market it is possible to estimate the relationship between the risk-neutral expected loss \( ELR_{i,T}^{PL} \) and the real-world expected loss of a performing loan \( EL_{i,T}^{PL} \). The best way to visualize this relationship is to build a scatter plot between the real-world \( EL_{i,t}^{PL} \) and \( ELR_{i,t}^{PL} \) implied from the market’s quotes. Since this relationship is time dependent, in order to be able to compare quotes on loans having

\[ \text{This is in line with finding of Bakshi et al. (2003) that theoretically investigated whether risk-neutral expected recovery rates are lower or higher than its physical (real-world) counterpart.} \]
different maturities, in the chart of figure 1 has been calculated the relationship by projecting the values of $EL_{i,t}^{PL}$ and $ELR_{i,t}^{PL}$ for $t = 1$.

![ELR vs EL - Syndicated March 2017](image)

Figure 1 - ELR vs EL (for a portfolio of syndicated loans issued between March/16 and March/17)

The red line in the chart of figure 1, represents the fitting obtained by using the following *bridge* function:

$$ELR_{i,t} = \Phi \left[ \Phi^{-1} \left( \frac{LGD_{i,t} \cdot CD{P^P}_{i,t}}{\lambda_{i,m}} \right) + \lambda_{i,m} \sqrt{t} \right]$$

(4.11)

while the small blue circles represent the performing $ELR_{i,t}^{PL}$, calculated using the following expression:

$$ELR_{i,t}^{PL} = \Phi \left[ \Phi^{-1} \left( \frac{LGD_{i,t} \cdot CD{P^P}_{i,t}}{\lambda_{i,m}} \right) + \lambda_{i,m} \sqrt{t} \right]$$
where \( \lambda_i \) was estimated to match the transaction prices \( TP_j \) observed on the primary syndicated market, using the following pricing function\(^\text{17}\):

\[
TP_j = \sum_{j=1}^{n(T)} CF_i(t_j) e^{-r(t_j)T} \left[ 1 - LGD_{i,j} \Phi \left( \Phi^{-1} \left( CDPR_{i,j} \right) + \lambda, \sqrt{T} \right) \right]
\] (4.12)

The parameter \( \lambda_{s,m} \) represent the systematic sharpe-ratio\(^\text{18}\), that is, the credit risk-premium observable on the debt market for loans belonging to the same cluster \( (s) \), that the investors are asking as a compensation of the risk of estimation of real-world expected loss.

The calibration of parameter \( \lambda_{s,m} \) can be performed with loans having a value of \( PD \) at 1-year below 90%. Above this value the borrower is almost in default and the equation that describes better

\[ \text{ELR} = \Phi \left( \Phi^{-1}(EL) + \lambda \right) \]

\[ \text{ELR[NPL]} = EL + (1 - EL) \left[ \Phi \left( \frac{\lambda}{2} \right) - \Phi \left( -\frac{\lambda}{2} \right) \right] \]

Figure 2 - Bridge function example (\( \sigma = \lambda/(1 - EL), \lambda = 0.3 \))

The calibration of parameter \( \lambda_{s,m} \) can be performed with loans having a value of \( PD \) at 1-year below 90%. Above this value the borrower is almost in default and the equation that describes better

\(^{17}\) See Tilloca (2015) for more details on the loans pricing using the proposed approach.

\(^{18}\) In this example the calibrated \( \lambda_{s,m} \) is equal to 0.338 and the expected loss has been calculated using the real-world default probability at 1 year. In this paper the sharpe-ratio is estimated directly from the primary loans market that is represented by the syndicated loans transactions available in Bloomberg. This methodology in an extension of the approach proposed in Berg and Kaserer (2008) and Berg (2010).

\(^{19}\) The estimated \( \lambda_{s,m} \), calculated using the equation (4.11), is close to the weighted average estimate of \( \lambda_i \), obtained using \( EL_i \) as the weight factor. The cluster \( (s) \) can be defined at a level of seniority or type of instrument or both. A more precise result could be obtained by defining clusters of loans that are homogeneous with respect to the recovery risk.
this situation was obtained in the section §4.1 and for a non-performing loan (NPL) the risk-neutral expected loss is given by the equation:

\[ ELR_{i,t}^{NPL} = EL_{i,t}^{NPL} + (1 - EL_{i,t}^{NPL}) \cdot \left\{ \Phi \left( \frac{1}{2} \sigma_{R_i} \cdot \sqrt{t} \right) - \Phi \left( -\frac{1}{2} \sigma_{R_i} \cdot \sqrt{t} \right) \right\} \]  
(4.13)

where \( t \) represent the expected total recovery time. In the Annex §A.1 is shown that in this case it is possible find an appropriate \( \lambda_{s,m} \) defined as:

\[ \lambda_{s,m} \approx (1 - EL_{i,t}^{NPL}) \cdot \sigma_{R_i} = R_{i,t}^{NPL} \cdot \sigma_{R_i} \]  
(4.14)

such that \( ELR_{i,t} \approx ELR_{i,t}^{NPL} \). The equation (4.14) allows the estimation of the market implied volatility \( \sigma_{R_i} \) necessary to evaluate the recovery risk-premium:

\[ \sigma_{R_i} \approx \frac{\lambda_{s,m}}{R_{i,t}^{NPL}} \]  
(4.15)

The equation (4.14) establishes a link between the market’s risk-premium \( \lambda_{s,m} \) and the volatility \( \sigma_{R_i} \) of the recovery process, so in the context of NPLs evaluation \( \lambda_{s,m} \) can be considered as the additional risk-premium that the investors are asking as compensation for the uncertainty on the recovery process and the consequent uncertainty of expected loss estimation.

It is possible to conclude that, in the case that the relationship (4.15) is satisfied, the following equation:

\[ ELR_{i,t} = \Phi \left[ \Phi^{-1} \left( EL_{i,t} \right) + \lambda_{s,m} \sqrt{t} \right] , \quad \text{with} \quad \begin{cases} EL_{i,t} = LGD \cdot CDP_{i,t}^D, & \text{if } CDP^D < 1 \\ EL_{i,t} = LGD_{i,t}, & \text{if } CDP^D = 1 \end{cases} \]  
(4.16)

can be considered as a bridge function between the performing and the non-performing loans since the same equation can approximate the correct behavior on both the regions. The meaning of \( ELR_{i,t} \) will be clearer by looking at the chart in figure 2 where it is possible to see an example on how the bridge function works. For values of \( \lambda \) in a range of 0.30-0.35, the bridge function represents a good approximation of the recovery risk option in a range of \( EL \) between 20% and 70% (where are located most of NPLs). The accuracy increases when the value of \( \lambda \) decreases and for \( \lambda = 0 \) the two equations provide the same estimate.
The region with $EL < 20\%$ includes the secured loans with a high level of quality of collateral and is in overlapping with the region of the performing loans, where the risk-neutral expected loss can be described by equation (4.11). The prices obtained by using the equation (4.13) will be lower if compared to the prices obtained using the (4.11), theoretically both the equations could be used to provide an estimate of the recovery risk-premium in this region. As reported by Jacobs (2012) the "realized" yields on defaulted debt is significantly higher for secured loans as compared to unsecured loans (about 65% more in relative terms), this means that the defaulted secured loans are mispriced by the market. If the investors were rationale it could be systematically convenient to sell unsecured debt and buy secured debt so that the difference on the "realized" yield will be almost eliminated. This is not what is observed on the defaulted debt transactions and the reason for such odd behavior could be due to the distressed market conditions in which these transactions take place. This is not what is expected to observe when both counterparts are professional investors and for orderly transaction\footnote{The IFRS-13 definition of orderly transaction is the following: A transaction that assumes exposure to the market for a period before the measurement date to allow for marketing activities that are usual and customary for transactions involving such assets or liabilities; it is not a forced transaction (e.g. a forced liquidation or distress sale).} as should be the selling of NPLs. The existence of the mispricing implies that we are dealing with an incomplete market and in such case we don't have a unique risk-neutral measure and is necessary.
the observation of existing trades to select the most appropriate one. In this case given the direction of the mispricing reported by Jacobs it is possible to conclude that the risk-neutral measure which is able to better represent a “fair” evaluation is the one observed on the performing loans that are not subject to the distressed market conditions often observed on the defaulted debt. The accuracy of the bridge function above the region of $EL > 70\%$ does not represent a big issue since in any case the value of the loan is already hard hit and a bank will be unwilling to sell a loan to a price near to zero since it has the option to cancel the debt.

The fact that the bridge function is able to explain the credit risk in both worlds, allows to extend to the NPLs, the market risk-premium observed on the performing loans. As example in figure 3 is shown the extension of the relationship between $EL(1)/ELR(1)$ to the region of NPL. The green triangles represent few NPL transactions observed on the market where all the necessary information were available\(^{21}\). This observation allows to conclude that the investors on NPLs are asking at least the same risk-premium observed in the loans primary market and that the bridge function represents the equation that best represents the market conditions at which it is expected that the NPL transactions will occur. Moreover, the validity of eq. (4.16) allows to conclude that the bank’s estimate of expected loss is reliable and that the market risk-premium asked by the investors is mainly driven by uncertainty of estimate of the EL.

---

\(^{21}\) The green triangles represent transactions of large portfolios of NPLs sold by Unicredit in 2016-2017, evaluated by projecting the market’s $ELR$ for $\tau = 1$ in order to be able to compare both performing loans having different maturities and NPLs having different total recovery time $\tau$. The red line represent the best fitting obtained using the equation (4.16) with $\tau =1$.
It is possible to extend the definition of standardized risk-premium $\lambda_{s,m}$ to also incorporate the specific risk (that can be considered as the risk-premium due to specific risk factors, see Chong et al. (2003)). To do that we define the observed risk-premium as the sum of market's risk-premium $\lambda_{s,m}$ and the transaction specific risk-premium $\xi_i$, as follows:

$$\lambda_i = \lambda_{s,m} + \xi_i \quad (4.17)$$

While the market's risk-premium can be estimated using a broad set of debt market instruments, $\xi_i$ can be estimated only if specific transactions are available. The chart of figure 4 shows the historical value of $\lambda_{s,m}$ calibrated from the primary loans market$^{22}$. It is possible to historically see the market spread having moved between a range of 2%-4% (average 3%) while the lambda market moved in a range between 0.25 to 0.35 (average 0.3). This is a good result because theoretically it is expected that this parameter should move slowly since most of the volatility should be captured by the rating and recovery expectations that are moved out from the estimation of this parameter. This result is in line with the analysis available in Berg and Kaserer (2008) where was found, for the 2004-2007 period, a market sharpe-ratio for Europe ranging from 0.26 to 0.28.

Given the relationship (2.3) it is possible show that the bridge function $ELR$ can be expressed also as a function of the recovery $R_{i,T}$ (see Annex §A.2 for a proof).

$$RR_{i,T} = \Phi \left( \Phi^{-1}(R_{i,T}) - \lambda_i \sqrt{T} \right) \quad (4.18)$$

the above relationship allows to conclude that the market risk-premium has the effect of reducing the value of real-world recovery estimate. This effect is complementary to that observed on the real-world expected loss. Since the market price of a non-performing loans represent the discounted estimate of the risk-neutral ultimate recovery, the equation (4.18) will be used in the next section to calculate the risk-neutral fair value of a non-performing loan.

4.3 The Fair Value NPL formula

It is a market practice to price the defaulted instruments after default in accordance with the expected future cash flows, that in the case of a private bank loan these are the workout cash-flows. As explained previously, the correct pricing of a defaulted loan requires that an investor estimates potential future recoveries, as well as the timing of them, and then discounts the expected cash-flows using the proper discount rate, which includes the required recovery risk premium. On the market, it

$^{22}$ To better highlight the trend the estimates are performed using the transactions observed in the last 12 months respect to the reference date.
usual that only the total return on the defaulted debt/loan is observable and because even the timing of the recoveries is subject to some degree of uncertainty, the proposed evaluation formula will follow the approach defined by the equation (2.8) so that the loan fair value will depend only by the expected ultimate recovery $R_{i,T}$. The non-performing loans have some features that are not available in the performing loans. The most important aspect that we consider is necessary to include in the model for evaluating the NPLs, is the default vintage effect, which represent a risk-neutral adjustment necessary to avoid temporal arbitrages. This effect is often ignored on the real-world models\textsuperscript{23} for the estimation of the economic LGD, but it is of significant importance for investors. As reported by Schuermann (2004) the time spent in bankruptcy can dramatically reduce the value of the recovery and this statement is even more true when is applied to the NPLs transactions where are often sold loans that have a default vintage greater than 1 year\textsuperscript{24}. This means that after an exposure is classified as bad loan all the attempts made by the bank to allow the borrower to come back into compliance with its obligations are failed, and for this reason the loan is considered by the investors riskier. To take into consideration the increase of recovery risk due to the default vintage the proposed model will use the effective recovery time $\tau$ to define the time-horizon for the recovery risk embedded in the recovery process. Since the risk-neutral recovery rate $RR_T$ can be considered as the market’s value of the recovery, it is possible to use the equation (4.18) to extend the use of (2.8) so that the fair value of a non-performing loan can be calculate as:

$$FV_i = e^{-\tau(\lambda_i)^{\frac{1}{\mu_i}}} \cdot N_i \cdot \left[ 1 - \Phi \left( \Phi^{-1} \left( EL_{i,T} \right) + \lambda_i \cdot \sqrt{\tau} \right) \right]$$

(4.19)

or equivalently as:

$$FV_i = e^{-\tau(\lambda_i)^{\frac{1}{\mu_i}}} \cdot N_i \cdot \Phi \left( \Phi^{-1} \left( R_{i,T} \right) - \lambda_i \cdot \sqrt{\tau} \right)$$

(4.20)

with:

\textsuperscript{23} Non-arbitrage conditions that must be considered in a risk-neutral evaluation, are not applied to real-world estimates.

\textsuperscript{24} According to EBA, the European Banking Authority the three subcategories of NPLs are: 1) overdrawn and/or past-due exposures; 2) unlikely-to-pay exposures; 3) bad loans. More specifically:

- **Bad loans** are exposures to debtors that are insolvent or in substantially similar circumstances.
- **Unlikely-to-pay** exposures (aside from those included among bad loans) are those in respect of which banks believe the debtors are unlikely to meet their contractual obligations in full unless action such as the enforcement of guarantees is taken.
- **Overdrawn and/or Past-due exposures** (aside from those classified among bad loans and unlikely-to-pay exposures) are those that are overdrawn and/or past-due by more than 90 days and for above a predefined amount.

Usually after the first payment default is necessary more than 90 days before the exposure is classified Past-due, then if borrower will continue to not meet its contractual obligations the exposure is classified unlikely-to-pay and finally if the borrower continues to remain in default the exposure is classified as Bad loan. Usually a bank takes about 1-year to classify an exposure as bad loan.
\[ \tau = t_m + \alpha \theta \]  \hspace{1cm} (4.21) 

and

\[ \lambda_i = \lambda_{s,m} + \xi_i \]  \hspace{1cm} (4.22) 

where \( \lambda_i \) is the total risk-premium, \( t_m \) is the weighted average recovery time as defined in (2.6) and \( \alpha \) the leverage of the default vintage \( \theta \). The estimation of \( \lambda_i \) requires the estimation both of systematic component \( \lambda_{s,m} \) and idiosyncratic component \( \xi_i \). But thanks to the bridge function introduced in the previous section, in the case of lack of observability of NPL’s market risk-premium, it is possible to use \( \lambda_{s,m} \) as a reasonable surrogate of \( \lambda_i \). The intuition behind this simplification is the diversification, which essentially eliminates the idiosyncratic risk in a portfolio where there are no significant large exposures. In a fine-grained portfolio, the idiosyncratic component \( \xi_i \) should be averaged out, consequently the total portfolio market risk-premium should be described by the systematic component\(^25 \) \( \lambda_{s,m} \).

![Effective Recovery Diagram](image)

**Figure 5 - The Default Vintage Effect**

In the proposed model the default vintage \( \theta \), introduced in (4.21), is added to the expected recovery time \( t_m \) to get the effective recovery time \( \tau \). In this way the model is able to consider also the recovery timing risk. To better understand the meaning of this parameter, we consider the simple case, shown in figure 5, where the bank expect a single recovery cash flow \( R_T \) at time \( T \) on a Notional equal to 1. Under this assumption, and assuming a zero risk-free rate and a constant \( \lambda \), an investor willing to buy the loan at time \( t_1 \), will evaluate the fair value as:

\(^{25}\) It was possible to reach this good approximation with the \( \lambda_{s,m} \) estimated on the performing loans by incorporating in \( \tau \) the default vintage effect risk component, which is not observable on the performing loans. So that on NPLs \( \lambda_{s,m} \) can be considered the systematic risk-premium net of default vintage risk.
then the investor could decide to sell the same loan later, at time $t_n$. In this case the buyer will evaluate the fair value as:

$$FV(t_n) = \Phi\left( \Phi^{-1}(R_T) - \lambda \cdot \sqrt{\tau_n} \right)$$

Since in this simple case, the time value of money is zero and it is missing any contractual obligation to respect the recovery time $T$, there is no reason for the first investor to make a profit on this transaction. In the absence of contractual obligations on the recovery time, temporal arbitrage are not allowed in a risk-neutral world\textsuperscript{26}, and so the two fair value must be the same. To be the same $\tau_1$ must be equal to $\tau_n$ and for this to happen it is necessary to include in $\tau$ the default vintage $\theta$.

However, it is also necessary to consider the possibility that the second investor may have a different opinion on the correctness of the estimate of $T$ and for this reason in the equation (4.21) has been added the parameter $\alpha$ that has the function of allowing an increase or decrease of the default vintage effect. In this context the parameter $\alpha$ could be used as an additional calibration parameter to better fit the observed market prices\textsuperscript{27}.

Usually with the passage of time, the weighted average recovery time $t_m$ should decrease, while $\theta$ should increase and so, on average, $\tau$ should remain constant, but it could happen that to recover a credit, the bank will spend more time than expected. In this case $\tau$ will increase and will increase accordingly also the recovery risk, this side effect is what we identify as the default vintage effect. In this model, the implied recovery risk option, embedded in the recovery risk-premium, expires only when the recovery becomes zero or when the remaining debt is canceled. In this context, it is possible to consider the option embedded in the model as a special case of perpetual option.

It is possible write the equation (4.20) using the theoretical spread to discount the expected recoveries, in such case the NPL fair value can be calculated using the eq.(2.9) by defining the following relationship:

$$RR_{1,T} = R_{1,T} \cdot e^{-\lambda(t_m)} \cdot \Phi\left( \Phi^{-1}(R_{1,T}) - \lambda \cdot \sqrt{t_m + \alpha \cdot \theta} \right)$$

\textsuperscript{26} In this example the value of $R_T$ has been kept constant, assuming that it is not subject to obsolescence. The temporal risk-neutral adjustment, introduced by the $\theta$ parameter, is applied on top of the real-world estimate, which could in turn be affected by a positive or negative vintage adjustment.

\textsuperscript{27} It has been observed that two loans having the same EL and the same expected recovery time $t_m$, but different default vintage $\theta$ is usually evaluated differently by the investors, the older loan being evaluated significantly lower in respect to the young one. An example of this behavior is shown in the Table 1, where the portfolio “Legacy” and “SME Unsec (Tranche 3)”, despite having similar expected recovery, they have a default vintage and a very different market price.
so that, the spread implied by using the equation (2.9) is given by:

\[
\sigma_s(t_m) = -\frac{1}{t_m} \ln \left( \frac{\Phi^{-1}(R_{s,T}) - \lambda_s \sqrt{t_m + \alpha \theta}}{R_{s,T}} \right)
\]  

(4.24)

**Vintage effect on Spread at 2.5 years**

The complexity of the expression for the spread \( s(t) \) explains why the estimation of this parameter using methods usually applied to liquid market is difficult and unstable. The equation (4.24) explains that, even in the simple situation where the spread observed for a given loan, is applied to a NPL having the same recovery, the spread could be different if the other NPL has a different vintage and/or expected recovery time. The problem of applicability of spreads to loans having different expected recovery is even more complex. The chart in figure 6 shows the effect of the default vintage \( \theta \) (between 1 and 6, and \( \alpha = 1 \)) on the spread at 2.5 years \( (t_m = 2.5) \). As it is possible to see, the impact of the default vintage \( \theta \) is quite significant: for an NPL having EL equal to 40%, the implied spread for a recovery time of 2.5 years change from 18% to 32%.
4.4 Empirical Results

As examples of application of the fair value formula (4.20), there will be considered the following NPL transactions:

<table>
<thead>
<tr>
<th>Portfolio Name</th>
<th>Loans Count</th>
<th>Expected Recovery</th>
<th>Default Vintage (Theta)</th>
<th>Expected Recovery Time</th>
<th>Market Price (%)</th>
<th>Implied Lambda</th>
<th>Lambda Market</th>
<th>Idiosyncratic Lambda</th>
</tr>
</thead>
<tbody>
<tr>
<td>Large Names</td>
<td>4,427</td>
<td>40.70%</td>
<td>2.85</td>
<td>2.58</td>
<td>12.55%</td>
<td>0.39</td>
<td>0.30</td>
<td>0.09</td>
</tr>
<tr>
<td>Legacy Old Loans</td>
<td>157,539</td>
<td>27.25%</td>
<td>12.26</td>
<td>1.87</td>
<td>3.15%</td>
<td>0.33</td>
<td>0.30</td>
<td>0.03</td>
</tr>
<tr>
<td>SME Secured</td>
<td>20,961</td>
<td>57.99%</td>
<td>3.71</td>
<td>2.15</td>
<td>25.10%</td>
<td>0.36</td>
<td>0.30</td>
<td>0.06</td>
</tr>
<tr>
<td>SME Unsec (Tranche 1)</td>
<td>153,057</td>
<td>39.30%</td>
<td>4.52</td>
<td>1.74</td>
<td>14.35%</td>
<td>0.32</td>
<td>0.30</td>
<td>0.02</td>
</tr>
<tr>
<td>SME Unsec (Tranche 2)</td>
<td>7,896</td>
<td>33.12%</td>
<td>1.68</td>
<td>2.88</td>
<td>13.96%</td>
<td>0.30</td>
<td>0.30</td>
<td>0.00</td>
</tr>
<tr>
<td>SME Unsec (Tranche 3)</td>
<td>12,935</td>
<td>29.84%</td>
<td>2.73</td>
<td>1.84</td>
<td>13.56%</td>
<td>0.27</td>
<td>0.30</td>
<td>0.03</td>
</tr>
</tbody>
</table>

Table 1 - Summary table using Unicredit transactions executed in 2016-2017

In the table above was used an average value for $\lambda_{s,m}$ of 0.3, and the idiosyncratic lambda $\xi_i$ has been calibrated to match the observed market prices. As it is possible to see the equation (4.20) is able to provide a good fit to observed prices, since $\xi_i$, that represent the specific variability premium (the component of $\lambda_i$ that cannot be explained by the systemic market risk) is quite low. In effect only the “Large Names” portfolio has a specific lambda that could be considered having a value quite significant, this value could be due to the fact that this portfolio cannot be considered fine-grained and so there is missing, the diversification effect that allows to reduce the idiosyncratic lambda. Once the default vintage $\theta$ has been taken into account using $\alpha = 1$, the implied lambda of the total portfolio shown in table 1 is about 0.29, confirming the observations performed on the syndicated loans market.

<table>
<thead>
<tr>
<th>Defaulted Debt Type</th>
<th>Return on Defaulted Debt (RDD)</th>
<th>LGD at Default</th>
<th>Market Price</th>
<th>Time to Resolution</th>
<th>Theta</th>
<th>Ultimate Recovery</th>
<th>RiskFree Rate</th>
<th>Implied Lambda</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term Loans</td>
<td>29.19%</td>
<td>55.46%</td>
<td>44.54%</td>
<td>1.64</td>
<td>1.00</td>
<td>67.79%</td>
<td>5.00%</td>
<td>0.31</td>
</tr>
<tr>
<td>Bonds</td>
<td>26.44%</td>
<td>56.37%</td>
<td>43.63%</td>
<td>1.32</td>
<td>0.10</td>
<td>59.47%</td>
<td>5.00%</td>
<td>0.27</td>
</tr>
<tr>
<td>Revolvers</td>
<td>25.88%</td>
<td>53.32%</td>
<td>46.66%</td>
<td>1.32</td>
<td>0.10</td>
<td>63.25%</td>
<td>5.00%</td>
<td>0.29</td>
</tr>
<tr>
<td>Loans</td>
<td>32.21%</td>
<td>52.81%</td>
<td>47.19%</td>
<td>1.25</td>
<td>1.00</td>
<td>66.90%</td>
<td>5.00%</td>
<td>0.29</td>
</tr>
<tr>
<td>Total Debt</td>
<td>28.56%</td>
<td>55.05%</td>
<td>44.95%</td>
<td>1.58</td>
<td>0.60</td>
<td>66.85%</td>
<td>5.00%</td>
<td>0.32</td>
</tr>
</tbody>
</table>

Table 2 - Summary table using Jacobs (2009) data at page 34 (Table 11)

It is possible to benchmark the proposed model with the data made available in Jacobs (2009), in this case the equation (4.20) can be applied to estimate the implied market risk-premium $\lambda_i$. The results of this estimation are shown in the table 2.

In the table above the ultimate recovery $R_{i,T}$ was estimated using the following equation:

---

28 The portfolios has been taken from Unicredit's NPLs transactions executed in 2016-2017.
29 This portfolio contains single names with large exposures.
\[ R_{i,t} = MP_{i,0}^t (1 + RDD)^T, \text{ with } MP_{i,0}^t = 1 - LGD_{i,\text{AtDefault}} \tag{4.25} \]

and the implied \( \lambda_i \) using the following equation:

\[ \lambda_i = \frac{\Phi^{-1}(R_{i,t}) - \Phi^{-1}\left( MP_{i,0}^t (1 + r_f)^T \right)}{\sqrt{T + \theta}} \tag{4.26} \]

In this case for the loans \( \theta = 1 \) was set, which is a common default vintage for the banks’ loans, while for Bonds and Revolvers \( \theta \) was set at about 1-month. As it is possible to see even in this case the implied market risk-premium is near to the average value of \( \lambda_{i,m} = 0.3 \) observed on the syndicated loans. This fact confirm the previous observation than on large portfolios the diversification, is able to eliminate the idiosyncratic risk so that, once the default vintage risk has been properly evaluated, the main component of the market risk-premium is represented by systematic lambda \( \lambda_{i,m} \).

<table>
<thead>
<tr>
<th>Business Unit</th>
<th>Obligor Count</th>
<th>Average Time</th>
<th>Discounted LGD at 15%</th>
<th>Ultimate Recovery</th>
<th>Risk Free Rate</th>
<th>Implied Lambda</th>
<th>Theta</th>
<th>ELGD Volatility</th>
<th>Theoretical Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>Large Corporates (U.S.)</td>
<td>676</td>
<td>3.33</td>
<td>41.6%</td>
<td>93.0%</td>
<td>5.0%</td>
<td>0.29</td>
<td>1.00</td>
<td>30.9%</td>
<td>9.4%</td>
</tr>
<tr>
<td>Large Corporates (non-U.S.)</td>
<td>268</td>
<td>2.58</td>
<td>37.3%</td>
<td>89.9%</td>
<td>5.0%</td>
<td>0.30</td>
<td>1.00</td>
<td>33.2%</td>
<td>12.0%</td>
</tr>
<tr>
<td>Real Estate</td>
<td>719</td>
<td>2.23</td>
<td>42.0%</td>
<td>79.2%</td>
<td>5.0%</td>
<td>0.27</td>
<td>1.00</td>
<td>33.7%</td>
<td>16.3%</td>
</tr>
<tr>
<td>Emerging Markets</td>
<td>394</td>
<td>3.04</td>
<td>42.2%</td>
<td>88.4%</td>
<td>5.0%</td>
<td>0.31</td>
<td>1.00</td>
<td>35.6%</td>
<td>12.7%</td>
</tr>
<tr>
<td>Middle Market</td>
<td>1,264</td>
<td>2.15</td>
<td>40.3%</td>
<td>80.6%</td>
<td>5.0%</td>
<td>0.31</td>
<td>1.00</td>
<td>38.4%</td>
<td>18.3%</td>
</tr>
<tr>
<td>Private Banking</td>
<td>310</td>
<td>1.66</td>
<td>34.5%</td>
<td>82.6%</td>
<td>5.0%</td>
<td>0.32</td>
<td>1.00</td>
<td>38.3%</td>
<td>19.8%</td>
</tr>
<tr>
<td>Total</td>
<td>3,761</td>
<td>2.43</td>
<td>39.8%</td>
<td>84.5%</td>
<td>5.0%</td>
<td>0.30</td>
<td>1.00</td>
<td>35.4%</td>
<td>15.0%</td>
</tr>
</tbody>
</table>

Table 3 - Summary table using Araten (2004) data at page 30 (Table 3)

Using the same approach it is possible to benchmark the model using the data reported in Araten et al. (2004). In this case, in the paper was reported the volatility of economic LGD, calculated by discounting the LGD using an average market yield\(^{30}\) of 15%. So that, by considering this volatility as a good proxy for the market implied recovery volatility \( \sigma_R \), it is possible to use the equation (4.14) to calculate the implied lambda and model yield. The results of this estimation are shown in the table 3, where the annual compounding theoretical yield, was calculated using the following expression:

\[ y_i = \left( \frac{R_{i,t}}{\Phi\left( \Phi^{-1}(R_{i,t}) - \lambda_i\sqrt{T + \theta}\right)} (1 + r_f)^{-T} \right)^{1/7} - 1, \text{ with } \lambda_i = \sigma_{ELGD} \cdot R_{i,t} \tag{4.27} \]

\(^{30}\) Using an average risk-free rate of 5%, an yield of 15% is equivalent to a market spread over the risk-free rate of 10%. This spread is compatible with the “realized” spreads of 9.4% observed on defaulted bank loans in Brady.
It is noteworthy that by applying the model equations above, the theoretical yield on the total portfolio is equal to the expected yield of 15% reported in the same paper. The “realized” yields calculated in Jacobs (2009), are in some case, significantly higher than those calculated in Araten et al. (2004). For example for the bank loans, in table 2 is reported a yield of 32.2% (with $\lambda = 0.29$), while in table 3 was calculated an average yield of 15% (with $\lambda = 0.30$). The reason for such difference is not due to the implied market risk-premium, which in both cases moves in a narrow range of 0.27-0.32, but to the different market condition that in 2004 were more favorable than to those observed in Jacobs (2009), with a better ultimate recovery\(^{31}\) of 87%, compared to that reported in table 2 which is 67% and with an average recovery time of 2.43 years versus the 1.58 years reported in Jacobs (2009).

It is possible to conclude that, despite the big difference observed on the discount spreads/yields, the market implied risk-premium $\lambda$, calculated taking into consideration the default vintage $\theta$, is moving in a narrow range of 0.27-0.32. A similar range was also observed on the performing loans. Thus the observed NPLs transactions can be considered “fair” from the point of view of the risk-premium per unit of risk and consequently the average “realized” yields do not seem to be affected by the “stressed” market conditions often observed on defaulted debt.

This result can be better understood by remembering that the Sharpe-ratio represents the ratio risk/reward demanded on average by investors. Although it is not possible to transfer the expected return from one market to another, as different markets have different returns and risks, it is reasonable to assume that the Sharpe-ratio may be the same for different markets. Investors’ expectations converge on this ratio, in their search for investments that could be considered “equivalent”.

### 4.5 Extension to Past Due and Unlikely to Pay loans

The equation (2.4) can be used to calculate the fair value also for the Past-due (PD) and Unlikely-to-pay (UTP) loans. In this case it is necessary to consider the probability ($\omega^\square$) that the loan can return performing after some time. Assuming that this probability has been estimated, the fair value can be simply calculated as:

$$\text{FV} = \omega^\square \cdot \text{FV}_{\text{perf}} + (1 - \omega^\square) \cdot \text{FV}_{\text{npl}}$$

(4.28)

Where $\text{FV}_{\text{perf}}$ represent the fair value of the “recovered” performing loan. The probability ($\omega^\square$) could be estimated from transaction prices that are including such kind of loans. In the case that it is

\(^{31}\) It is noteworthy that an average ultimate recovery of 82%, for the bank loans, is reported also in Emery et al. (2007), confirming the better recovery conditions reported in Araten et al. (2004).
not possible to perform a calibration of this parameter using market prices, it is still possible to estimate the \textit{physical probability} $\omega^P$ using bank’s historical data, but in this case the reliability of fair value estimate will be affected by the fact that a \textit{real-world} estimate is used as substitute of a \textit{risk-neutral} one.
5. Conclusion

The most difficult activity related to the evaluation of NPL is the estimation of expected loss. But even in the situation that the EL estimate has been obtained using the best models available its estimate is subject to some degree of uncertainty. The proposed model assumes that the bank’s estimate of workout recovery is reliable and the market risk-premium asked by the investors is mainly driven by the recovery risk, explained by the unobservable parameter $\sigma_R$ of the recovery stochastic process $R(t)$ defined as an extension of the Merton’s process in the post-default state. Usually the investors cannot have the same statistical diversification and loan volumes that can lower the statistical uncertainty of the real-world estimate of recovery. It is reasonable that they are asking a premium for this. Even if a satisfactory model is missing that can explain the mechanisms of formation of this uncertainty premium, the proposed approach allows to measure it on the performing loans, so that can be used as a proxy for the unobservable parameter $\sigma_R$, thus allowing to adjust the estimation of the physical recovery. In this context the pricing model is able to provide good estimates especially when applied to large portfolios (where the idiosyncratic risk is low) and is requiring few parameters to work: the expected recovery ($R_T$), the expected recovery time ($t_m$), market risk-premium ($\lambda$) and the default vintage ($\theta$). The fact that the few observable NPL transactions, are pricing an uncertainty risk-premium (net of default vintage risk) that is near to the one observed on the performing loans market, means that the price offered by the investors seems to be reasonable. Before the development of an appropriate model for the pricing of NPLs was difficult for a bank to evaluate if the offered price was too far from a fair value estimate. So, a model that can help the bank to estimate a “fair” price for this kind of assets, which is not biased by the stressed market conditions often observed on defaulted debt, can help in reducing the difference between the banks’ asked price and the offered market price facilitating the development of a NPLs market.
A. ANNEX

A.1 ELR Approximations

In this section will be shown the equivalence between the following functions:

\[
\begin{align*}
ELR_{\text{NPL}} &= EL_t + (1 - EL_t) \left\{ \Phi \left( \frac{1}{2} \sigma_R \sqrt{t} \right) - \Phi \left( \frac{1}{2} \sigma_t \sqrt{t} \right) \right\} \\
ELR_i &= \Phi \left( \Phi^{-1}(EL_t) + \lambda_m \cdot \sqrt{t} \right) 
\end{align*}
\]

to do that it is necessary to write the Taylor’s expansion of \( \Phi(x) \) at about \( x_0 = 0 \), so that:

\[
\Phi(x) \approx \Phi(0) + x \cdot \Phi'(0) + \frac{1}{2} x^2 \cdot \Phi''(0) = \frac{1}{2} + \frac{x}{\sqrt{2\pi}}
\]

and

\[
\Phi^{-1}(x) \approx \sqrt{2\pi} \left( x - \frac{1}{2} \right)
\]

using the above approximation it is possible obtain the following equations

\[
\begin{align*}
ELR_{\text{NPL}} &\approx EL_t + (1 - EL_t) \frac{\sigma_R}{\sqrt{2\pi}} \sqrt{t} \\
ELR_i &\approx EL_t + \frac{\lambda_m}{\sqrt{2\pi}} \sqrt{t}
\end{align*}
\]

so that for

\[
\lambda_m = (1 - EL_t) \cdot \sigma_R
\]

or

\[
\sigma_R = \frac{\lambda_m}{R_t}
\]

the two expressions are equal.

The equation \( ELR \) can be seen as a bridge function between the performing and non-performing loans. More interesting the volatility \( \sigma_R \) is linked to systematic market risk-premium \( \lambda_m \), so that the market risk-premium observed on the performing loans can be seen as a measure of the uncertainty of the estimation of the recovery.
A.2 ELR and RR relationship

In this section will be shown the equivalence between the following functions:

\[
\begin{align*}
ELR_{npl}^t &= 1 - \Phi^{-1}(EL_{npl}^t) + \lambda_{npl} \sqrt{t} \\
RR_{npl}^t &= \Phi^{-1}(RR_{npl}^t) - \lambda_{npl} \sqrt{t}
\end{align*}
\]

In effect by using the Taylor’s expansion (A.1) and (A.2), it is possible to write:

\[
1 - \left(1 - R_t + \frac{\lambda_{npl} \sqrt{t}}{\sqrt{2\pi}}\right) \equiv R_t - \frac{\lambda_{npl} \sqrt{t}}{\sqrt{2\pi}}
\]

and hence

\[
R_t - \frac{\lambda_{npl} \sqrt{t}}{\sqrt{2\pi}} \equiv R_t - \frac{\lambda_{npl} \sqrt{t}}{\sqrt{2\pi}}
\]

so that it is possible to define the bridge function for the risk-neutral recovery rate \(RR\) as:

\[
RR_{npl}^t = \Phi^{-1}(RR_{npl}^t) - \lambda_{npl} \sqrt{t} \quad (A.4)
\]

the above relationship allows to conclude that the market risk-premium has the effect of reducing the value of real-world recovery estimate.
References


