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A.Bertagna, D.Deliu, L.Lopez, A.Nassigh, M.Pioppi, F.Reffel, P.Schaller, R.Schulze

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Internal Default Risk Model: Simulation of Default Times and Recovery Rates within the new FRTB framework

Andrea Bertagna∗1, Dragos Deliu∗2, Luca Lopez∗1, Aldo Nassigh∗1, Michele Pioppi∗†1, Fabian Reffel∗†3, Peter Schaller∗†3, and Robert Schulze∗2

1 UniCredit SpA, Piazza Gae Aulenti 3, 20154 Milano, Italy
2 UniCredit Bank Austria AG, Julius Tandler Platz, 1190 Wien, Austria
3 UniCredit Bank AG, Arabellastraße 12, 81925 Munich, Germany

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Abstract

In January 2016 the Basel Committee of Banking Supervision published the new requirements for the calculation of market risk within the banking sector. These requirements go under the name of Fundamental Review of the Trading Book. The default risk model is one part of these requirements which is subject to material changes: recovery rates must be stochastic variables, basis risk due to differences in recoveries have to be considered, and a dependence between recovery rates and systematic risk factors used to simulate default times must be enforced. This paper presents a new default risk model for market risk that is consistent with these requirements. The recovery rates follow a waterfall model which is based on a minimum entropy principle. Moreover, the model features correlation between default times and stochastic recovery rates by exploiting the observed correlation between default frequency and average recoveries in historical data. Besides setting the mathematical background, numerical results and impacts of the various model parameters are presented. These show, that the introduced correlation can have a significant impact on the capital charge.

Keywords: Banking Regulation; Market Risk; Fundamental Review of the Trading Book (FRTB); Default Risk Charge; Stochastic Recovery Rates; Rank Correlation.

Key messages:
New default risk model consistent with the Fundamental Review of the Trading Book;
Model features a waterfall model for recovery rates based on minimum entropy principle;
Model features correlation between default times and stochastic recovery rates;

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†Corresponding authors; Michele Pioppi: Michele.Pioppi@unicredit.eu, Fabian Reffel: Fabian.Reffel@unicredit.de, Peter Schaller: Peter@ca-risc.co.at.
1 Introduction

In January 2016 the Basel Committee on Banking Supervision (2016) revised the standards for the minimum requirement for Market Risk (Fundamental Review of the Trading Book, hereafter FRTB) that will replace the current framework known as Basel 2.5. Three years later, in January 2019, the Basel Committee on Banking Supervision (2019) published the final framework which serves as a basis for the various jurisdictions around the world for their local transposition into law. All affected banks in the respective jurisdiction have to comply with these rules and thus preserve a certain capital buffer for positions held on the books.

The framework consists of two parts: a mandatory Standardised Approach (SA) based on regulatory prescribed rules, and an internal models approach (IMA) which allows banks to use more sophisticated models provided they comply with a set of requirements. Moreover, a certain percentage of the capital requirement stemming from the SA serves as a floor for the capital charge coming from the IMA.

The IMA capital charge is made of three components: An Expected Shortfall model that is meant to cover losses arising from historical movements of market variables with a sufficient number of observations in the last year, a model that covers the remaining market variables and a Default Risk Charge (DRC). The DRC is meant to cover losses arising from issuers’ defaults, i.e. those losses that are not captured by the credit spread volatility in the Expected Shortfall model. The DRC will replace the current Incremental Risk Charge (IRC) that, besides defaults, also embeds losses coming from migration events. While the removal of rating migrations simplifies the calculations, the mandatory requirements from FRTB potentially complicate the model as compared to IRC. Specifically, the main requirements the DRC model must comply with are:

1. Besides credit products, the simulation must also include equity instruments;
2. Basis risks due to differences in recoveries for securities issued from the same obligor must be considered;
3. Maturity mismatches between a position and its hedge must be taken into account;
4. In addition to credit spreads, also equity spot prices must be used to determine default correlations;
5. Recovery Rates (RR) must be stochastic variables;
6. The internal default risk model must include the dependence between RR and systematic risk factors used to simulate default times;

This paper describes an algorithm for the DRC calculation that copes with these regulatory requirements, with a special focus on points number two, five and six. The other requirements are naturally embedded in our model but are not discussed in detail in this paper. Indeed, the Jump-To- Default (JTD) values for equities are simply included in the scope, the maturity mismatch is accounted for in the explicit simulation of default times, and default correlations are supposed to be external model inputs.

However, given its relevance, the calibration of default correlations (point four) deserves some discussion. Solutions to this problem can be classified according to two approaches, namely the default correlation can be derived via maximum likelihood (ML) from the time series analysis of past defaults on representative portfolios, as recently performed in Chava et al. (2011). An extensive survey of methodologies based on ML estimation is given in Altman (2008). Opposite to this approach, default correlations can be derived from market data, following the methodology first introduced in the nineties with CreditMetrics © by Gupton et al. (1998).

Derivation of the default correlation via ML has the obvious advantage of being the most straightforward way to infer from time series analysis the variable under consideration, i.e. the correlation among actual default. The disadvantage of this approach arises since default is - in general - a rare event. As a consequence, in order to be statistically significant, the ML estimation must be applied to a large sample of firms observed over a multi-year period. As a matter of fact, it is very difficult to build such a large database with proprietary data within a single financial institution and the analyst must resort to large, publicly available databases based on rating agencies’ data like, for instance, the Moody’s Ultimate Recovery Database¹ (Moody’s URD). However, usage of rating agency data is an effective way of deriving the loss distribution for credit portfolios whose issuers have access to

¹See: https://www.moodys.com/Pages/Default-and-Recovery-Analytics.aspx for more details
the capital market (for debt instruments), which are quite different from the typical credit portfolio of a bank, particularly in Europe. On top of that, the vast majority of default events recorded by rating agencies is related to sub-investment grade issuers, and this puts into question also the representativeness of the correlation estimations when applied to credit portfolios mainly exposed to investment-grade names, which is the case for portfolio subject to the DRC.

For the two reasons above, derivation from market data following CreditMetrics® is now widespread within the financial industry for Credit Value at Risk Models. Such approach has, in turn, the obvious disadvantage that default correlation is derived thanks to proxies: the correlation between equities (or CDS) as a proxy for asset correlations that, in turn, proxies default correlation.

Coming to the main points of the paper, the second requirement is achieved via a waterfall mechanism that accounts for three different seniorities. Requirements number five and six are the most demanding since they ask for a dependency between simulated defaults and recoveries. How to model this dependency is still an open topic that is subject to investigation among analysts from industry and academy. The text from the Basel Committee on Banking Supervision (2016) does not prescribe any explicit model but only requires banks to reflect the economic cycle within their loss estimates.

The first open issue in the debate concerns the economic reasons for the observed dependence between DF and LGD. This can be grounded in the fluctuations of the value of the firm asset used as collateral to pay back creditors after bankruptcy. Following this approach, Frye et al. (2000) and Frye (2000) developed a model in which the realized RR is driven by the same macro-factor which drives the firm’s Probability of Default (PD). This is a single-factor model, which extends the Asymptotic Single Risk Factor Model (ASRF) introduced by Gordy (2003) to the stochastic RR case and is now embodied in the risk weight formula of the Basel II regulation from the Basel Committee on Banking Supervision (2005).

As opposed to this approach, Altman et al. (2002) derive the dependency between DF and LGD on the basis of a demand-and-supply argument. Indeed, they observe that the market of defaulted debt lacks breadth. As a consequence, in years of high DF the market price of defaulted securities (the so-called market-recovery-rate or market RR) is depressed because of the increased offer. Under the latter approach, we should observe a lower dependency between DF and LGD when the workout-recovery-rate (workout RR) is considered, i.e. the value of the actual recoveries gathered by the holder of the defaulted debt in the months (or years) after the default, discounted at the time of default.

Workout RR is more difficult to estimate than market RR, and indeed most of the data published by rating agencies concern the latter. However Düllmann and Gehde-Trapp (2004) were able to compare the two measures and they found that the dependency between DF and LGD is much less important if workout RR is considered. They measured the correlation between DF and RR via ML on a representative portfolio of defaulted debt and found a less significant p-value and a smaller absolute value of the correlation. In particular they found that the hypothesis of non-zero correlation is significant at the 99% level only if the market RR is considered.

This latter result could explain the difference between the position taken by the Regulators regarding the minimum capital requirement for credit risk, i.e. the usage of a stressed LGD, and the requirement set forth for the DRC in the trading portfolio, namely the request for a full simulation. Economic losses for the credit portfolio of a representative bank will be driven by the price of the defaulted firm’s assets at the emergence of the recovery process. Opposite to it, the bank is expected to cash-in securities in the trading book in the aftermath of the default and therefore the economic loss will be determined by the level of the market RR.

The models developed by Frye (2000), Pykhtin (2004) and Düllmann and Gehde-Trapp (2004) belong to the single-factor family. However, the DRC must be calculated according to a multi-factor model, in continuity with the current best-practice for IRC models. Acharya et al. (2007) and Varma and Cantor (2005) analyzed the linear multi-factor dependence of creditor recoveries. Later, Chava et al. (2011) estimated the coefficients of alternative multi-factor models for the forecast of the loss distribution via ML on representative portfolios of defaulted debt. These models take into account both stochastic PD and stochastic RR. Their results confirm the materiality and statistical significance of a negative correlation between DF and market RR as already shown in the single-factor models and in the empirical analysis of Altman et al. (2005).

The algorithm proposed in this paper relies on the above results and is tailored for implementation, risk management and measurement purposes. It consists of four steps: In the first step, we define so-called fundamental risk factors that serve as systemic drivers. Then, correlated default times are

\[ LGD = 1 - RR. \]
simulated via a copula, e.g. Gaussian or Student’s t copula. In a third step, and RR are simulated and correlated to the default times. Finally, the data is combined with the JTDs.

The paper is organized as follows: Section 2 describes the four steps of the algorithm to simulate correlated default times and RR. Section 3 reports a numerical example. Conclusions are in Section 4. All the mathematical details of the algorithm are provided in the Appendix.

2 Algorithm

This section describes the steps of the algorithm underlying our DRC model. Mathematical details and additional information are provided in the Appendix.

2.1 Fundamental Risk Factors

First of all, we select a given number of predefined risk factor time series that are supposed to represent the whole market. A risk factor time series could be a credit spread for a given maturity of a specific company over a certain time period, e.g. the 5 year pillar for the Google CDS curve from 2017 to 2007 as a representative of tech companies. From these risk factor time series, non-overlapping normalized ten-day returns are calculated. The return calculation depends on the risk factor type: differences for credit risk factors and log-returns for equities. In order to have homogenous variables, returns are normalized by the standard deviation.

These normalized returns are used as explanatory variables in the regression on the individual issuer’s time series. In order to improve robustness and stability of the regression, we first perform a Principal Component Analysis (PCA) and select the first Q components that explain a sufficient portion of the variability of the original returns. These components represent the Fundamental Risk Factors (FRF) entering the model.

2.2 Simulation of Correlated Default Times

We now focus on the actual issuers to be simulated. Again, we determine for each issuer a risk factor time series and calculate the respective normalized returns. We then perform a linear regression of the normalized returns on the FRF with zero intercept. This gives us for each issuer \( j \) a set of coefficients

\[
(\beta_{jl}), \quad l = 1, \ldots, Q.
\]  

(2.1)

We model the default time \( \tau_j \in \mathbb{R} \) for the issuer \( j \) assuming an exponential distribution for the default times. The rate parameter is calibrated to the issuer’s one year PD. To model the dependence on the FRF a copula method is applied. A discussion about the use of different copulas (Gaussian, Student’s t, Clayton) and their impact on DRC is given in Wilkens and Predescu (2017). Their study reveals that for diversified portfolios, the capital impact of different copulas is around 10%. Our proposed method for the default time and RR simulation is not depended on the chosen copula, i.e. the copula can be exchanged easily with very little adaptions that do not affect other parts of the method than the default time simulation itself. To keep it simple, we select the Gaussian copula (For a discussion on the extension to a Student’s t copula see Appendix C). Therefore, we have

\[
\tau_j = \frac{1}{\log(1 - PD_j)} \log(\Phi(z_j))
\]  

(2.2)

where

\[
z_j = \sum_{l=1}^{Q} \beta_{jl} \epsilon_l + \sqrt{1 - \sum_{l=1}^{Q} \beta_{jl}^2 \tau_j},
\]  

(2.3)

\( \Phi \) is the cumulative standard normal distribution, \( PD_j \) is the one year PD for issuer \( j \), and \( \epsilon_l \) and \( \tau_j \) are independent normal standard variables. The Appendix C offers a more detailed explanation and motivation of the formulas.
2.3 Simulation of Recovery Rates

2.3.1 Waterfall Mechanism for Recovery Rates

Our waterfall model has three main characteristics. First, three different seniorities are taken into account. Second, the RRs are simulated according to a minimum entropy distribution. This distribution has two main features:

• density distribution is positive at zero and one, thus RRs close to the boundaries have a non-zero probability. This is in line with historical data.

• the minimum entropy distribution comes with a theoretical framework that allows to easily constraint the waterfall parameters.

Third, the dependence between defaults and RRs is induced via a rank correlation mechanism that can be calibrated to historical data which reveals that the number of defaults and LGD are usually positively correlated.

A more detailed discussion about the properties of the minimum entropy distribution to simulate RRs can be found in Schaller (2018).

2.3.2 Waterfall Model in Detail

Our model assumes four types of issues subject to different recoveries:

1. secured debt,
2. senior unsecured debt,
3. subordinated debt,
4. equity.

In addition, we assume that the recovery process can be generalized as follows:

• All secured issues have the same RR. This is e.g. the case, if they are linked to the same collateral pool and no subordination between secured issues is defined.

• There is only one level of subordination between unsecured issues.

• Subordinated debt has a nonzero RR only if senior debt fully recovers.

• Equity always yields zero RR as required by the Basel Committee on Banking Supervision (2016, §186(c)).

In the case of default of an issuer the RRs are linked by the following waterfall mechanism. The collateral allocated to the secured debt is used to pay back the secured debt. The remaining secured debt is then treated as senior unsecured debt. The remaining assets of the issuer are then used to pay back the subordinated debt. After all of the senior unsecured debt has been payed back remaining assets are used to pay back the subordinated debt. Finally, the RR on the equity tranche is forced to be 0, although in practice, any assets still remaining after this step would be distributed to the equity holders. In case the collateral associated to the secured tranche is not enough to repay the secured debt, the assets associated to the unsecured tranche are used to also repay part of the outstanding secured debt. The RR of such portion of the secured debt is equal to the RR of the unsecured tranche.

The input of the model are the expected RRs $RR_{j_{\text{sec}}}$ per issuer $j$ and seniority $s\text{r} \in \{ \text{sec}, \text{sen}, \text{sub} \}$. Realizations of the RRs are denoted by small letters. According to our assumptions from above, the RRs are ordered such that $1 \geq r_{j_{\text{sec}}} \geq r_{j_{\text{sen}}} \geq r_{j_{\text{sub}}} \geq 0$ and so are the expected RRs. To get rid of this dependency, conditional expected RRs are used, i.e.

1. $P_{j_{\text{sec}}} \in [0; 1]$ as the expected portion of secured debt covered by the collateral;
2. $P_{j_{\text{sen}}} \in [0; 1]$ as the expected portion of senior unsecured debt covered by the remaining assets of the issuer conditional on the event that the subordinated debt has zero recovery;
3. $P_{j_{\text{sub}}} \in [0; 1]$ as the expected portion of subordinated debt covered by the remaining assets of the issuer conditional on the event that the senior unsecured debt has full recovery;
4. The event that the senior unsecured debt has full recovery has the probability $p^j \in [0; 1]$. 


The expected RRs are connected. The RR of the secured tranche is the portion of secured debt \( P^j_{sec} \) covered by associated collateral plus the RR of the senior tranche for the portion of the secured debt not covered by the collateral, i.e.

\[
RR^j_{sec} = P^j_{sec} + (1 - P^j_{sec})RR^j_{sen}.
\]  

(2.4)

The expected value of the RR for the senior unsecured tranche, \( RR^j_{sen} \), is the probability-weighted sum of both the event that the senior unsecured tranche is fully recovered and the conditional recovery of the senior tranche given that the remaining assets of the borrower do not suffice to fully recover the senior unsecured tranche, i.e.

\[
RR^j_{sen} = p^j \cdot 1 + (1 - p^j) \cdot P^j_{sen}.
\]  

(2.5)

The subordinated tranche recovers only in the event that the senior unsecured tranche fully recovers. Thus we have

\[
RR^j_{sub} = p^j \cdot P^j_{sub} + (1 - p^j) \cdot 0 = p^j \cdot P^j_{sub}.
\]  

(2.6)

We solve these equations for \( P^j_{sec} \) to calculate the mean of the conditional expected RRs as a function of \( RR^j_{sec} \) for \( s \in \{ sec, sen, sub \} \). Consequently, we need a fourth equation. This equation is retrieved from the minimum entropy principle, which is detailed out in the Appendix D. The resulting relation is

\[
p^j = \frac{RR^j_{sub}}{RR^j_{sub} - RR^j_{sen} + 1}.
\]  

(2.7)

Thus, we have four variables, \( p^j \), \( P^j_{sec} \), \( P^j_{sen} \), and \( P^j_{sub} \) and four equations 2.4, 2.5, 2.6 and 2.7. Solving these for \( P^j_{sec} \), \( P^j_{sen} \) and \( P^j_{sub} \) gives

\[
P^j_{sec} = \frac{RR^j_{sec} - RR^j_{sen}}{1 - RR^j_{sen}},
\]  

(2.8)

\[
P^j_{sen} = RR^j_{sen} - RR^j_{sub},
\]  

(2.9)

\[
P^j_{sub} = RR^j_{sub} - RR^j_{sen} + 1 = 1 - P^j_{sen}.
\]  

(2.10)

To simulate the realized recoveries \( r^j_{sec} \), \( r^j_{sen} \) and \( r^j_{sub} \) for issuer \( j \) and for the three different types of issues (senior secured, senior unsecured, subordinated) we need the following random variables:

1. A bivariate variable \( B^j \), taking the value 1 if there is a nonzero recovery for subordinated debt (i.e. with probability \( p^j \)) and 0 otherwise.
2. A variable \( V^j_{sec} \) with mean \( P^j_{sec} \) on the interval \([0; 1]\) for the portion of secured debt covered by the associated collateral.
3. A variable \( V^j_{sen} \) with mean \( P^j_{sen} \) on the interval \([0; 1]\) for the conditional recovery of the senior unsecured debt in the case it does not fully recover.
4. A variable \( V^j_{sub} \) on the interval \([0; 1]\) with mean \( P^j_{sub} \) describing the conditional realized recovery for the subordinated debt in the case it is nonzero.

From the minimum entropy principle, we can conclude that the variables \( V^j_{sub} \) and \( V^j_{sen} \) are distributed according an exponential distribution on \([0; 1]\) which is fully determined by its mean. Also \( V^j_{sec} \) is distributed according an exponential distribution on \([0; 1]\) as can be seen within the Appendix D.3 as a simplified case of \( V^j_{sub} \) and \( V^j_{sen} \).

The algorithm for the simulation of the RRs is then the following

\[
r^j_{eq} = 0
\]  

(2.11)

\[
r^j_{sub} = b^j v^j_{sub}
\]  

(2.12)

\[
r^j_{sen} = b^j + (1 - b^j) v^j_{sen}
\]  

(2.13)

\[
r^j_{sec} = v^j_{sec} + (1 - v^j_{sec}) r^j_{sen}
\]  

(2.14)

Thereby, \( b^j \) is a realization of the random variable \( B^j \), \( v^j_{sub} \) a realization of the random variable \( V^j_{sub} \) and analogously for \( v^j_{sen} \) and \( v^j_{sec} \).
2.3.3 Correlation between Default Times and Recovery Rates

If an economy is in a downturn, the defaulted companies cannot rely on other companies to take them over or on a government bailout. Hence, simultaneous defaults of multiple issuers usually go hand in hand with low RRs, especially for the unsecured tranches. In the introduction, we provided further explanations and referenced respective literature to underpin this observation. Therefore, we introduce a dependence of the DF with the unsecured RRs. More precisely, we correlate the number of defaults in a scenario with the sum of the senior unsecured LGD\textsubscript{sen}, and the subordinated LGD\textsubscript{sub}. This dependence is achieved via a rank correlation between the number of issuers defaulting before one year and the simulated LGDs. This rank correlation $\rho$ is induced via an order map $d_j^s$ that is used to align the ordering of the number of defaults in scenario $s$ to the ordering of the sum of the respective LGDs for that issuer $j$. Details on the method to compute a rank correlation between two distributions are given within the Appendix E.

The level of correlation $\rho$ is an exogenous parameter and affects significantly the order map. In particular, if $\rho = 1$, the map induces the exact same ordering between LGD and DF. This means that scenarios with the highest numbers of defaults are those with the highest aggregated LGDs. This represents the most punitive scenario for the bank. On the other extreme with $\rho = -1$, the map induces a reverse ordering giving rise to milder default losses. Any $\rho \in (-1; 1)$ does a mixtures of these both extreme variants and therefore establishes a correlation between the number of simultaneous default events and the level of LGDs. In the special case $\rho = 0$, DF and LGDs are uncorrelated.

2.3.4 Calibration of Rank Correlation

One of the most relevant aspects of the proposed model is the possibility to calibrate the issuer’s rank correlation $\rho$ introduced in Section 2.3.3 to historical data. Several databases of rating agencies store information of both default events and recovery rates over the past years. The former information is used to obtain the DF while the latter is used to compute the average LGD. The rank correlation $\bar{\rho}$ between DFs and the sum of the average LGD is quantified via the Spearman’s rank correlation coefficient. It is worth noting that the average LGD has a higher correlation with the DF than the LGD of the individual issuer. This can be inferred by considering the case of $n$ normally distributed variables. If $n$ variables $X_i$ have an independent idiosyncratic component and have a correlation $\rho_0$ with a given variable $Y$, i.e. $X_i = \rho_0 Y + \sqrt{1 - \rho_0^2} \epsilon$, then the sum $X = \sum_i X_i$ has a correlation $\rho_{\text{sum}}$ with $Y$ given by

$$\rho_{\text{sum}} = \frac{\mathbb{E}[XY]}{\sqrt{\mathbb{E}[X^2] \mathbb{E}[Y^2]}} = \frac{n\rho_0}{\sqrt{n^2 \rho_0^2 + n(1 - \rho_0^2)}}. \quad (2.15)$$

This framework is used to approximate the single issuer correlation $\rho$ given the correlation $\bar{\rho}$ between DF and average LGD as

$$\rho = \frac{\bar{\rho}}{\sqrt{\bar{\rho}^2 + n(1 - \bar{\rho}^2)}} \quad (2.16)$$

where $\rho_{\text{sum}}$ and $\rho_0$ are proxies of $\bar{\rho}$ and $\rho$ respectively and $n$ is the average number of yearly defaults.

2.4 Aggregation and Default Risk Charge Computation

For each scenario and for each instrument with at least one defaulting underlying issuer in that scenario, i.e. underlying issuers with default times lower or equal to the instrument’s maturity, a JTD value has to be calculated. The calculation depends on the instrument’s seniority and the simulated RR. The JTD in each scenario is given by the sum of the individual instruments JTDs. According to FRTB, the 99.9% quantile of the JTD distribution corresponds to the DRC.

3 Examples

The performances of the proposed simulation scheme are numerically assessed in terms of capital charge for a hypothetical portfolio. In Section 3.1 the exercise setup is described while in Section 3.2 results are reported and discussed.
3.1 Setup of the Example

3.1.1 Choice of Fundamental Risk Factors

According to the regulation two types of FRFs must be used. To comply with this rule, credit spread indices for different sector/rating combinations and country (or region) equity indices are used as FRFs. On top of the obvious difference between spread and equities, such typologies capture different source of correlations among issuers: spreads reflect the affiliation of issuers to the same sector/rating while equities are more related to the issuer region. In this example, from an initial list of 120 FRFs composed by around 40 equity indices from the most relevant markets and 80 generic credit spreads, about 50 components are selected via PCA. These components cover more than 99% of the system variance (see Figure 1).

![Explained variance vs. principle components](image)

Figure 1: The explained variance of the system vs. the number of selected components from the principal component analysis (PCA) is plotted with the 99% point of explained variance marked with red lines.

3.1.2 Choice of time interval

Regulation prescribes to use at least 10 years of time series to calibrate the model. For the choice of the time length $\delta$ for return calculation, 10 days is deemed a good compromise between regulation requests and accuracy in the estimation of regression parameters.

3.1.3 Choice of hypothetical portfolio of issuers

The hypothetical portfolio has been designed in such a way to have an overall exposure split in equal parts between corporate and sovereign. The JTD of corporates is dominated by equity while exposures to sovereigns are driven by unsecured senior bonds. It has the following peculiarities:

- 5000 corporate issuers
  - JTD of the corporates is linearly distributed between -10 and +12 million. If an issuer has a negative JTD value, the bank realizes a profit when the issuer defaults.
  - JTD of each corporate has the following composition:
    - 5% Secured debt
    - 15% Unsecured senior debt
    - 20% Subordinated debt
    - 60% Equity
• 100 sovereign issuers
  – JTD of the sovereign is linearly distributed between -50 and +150 million.
  – JTD of each corporate has the following composition:
    * 10% Secured debt
    * 80% Unsecured senior debt
    * 10% Subordinated debt
• The average correlation among issuer default times is slightly above 20%
• The average idiosyncratic component in the simulation of default times is 50%
• The default probability of each issuer in the first year is 0.5%
• The RR of each issuer for the equity component is 0.0
• The average RR of each issuer for the secured component of the debt is 0.8
• The average RR of each issuer for the senior unsecured component of the debt is 0.4
• The average RR of each issuer for the subordinated component is 0.2
• Each issuer reduces its JTD linearly in time with a decrease of 70% after the first year (Figure 2).

**Linear JTD decrease over time**

![Linear JTD decrease over time](image)

Figure 2: The JTD decreases linearly as a function of time with the JTD components for corporates shown on the left and the JTD components for sovereigns shown on the right.

The level of rank correlation between the DF and the average LGD is varied between 0% and 100% and results are computed for all the configurations.

### 3.2 Results and Discussion

The presence of a rank correlation between the DF and the LGDs affects the loss distribution of the bank. In Figure 3, three extreme cases ($\rho = -1, 0, 1$) are shown. As expected, when $\rho = 1$ the tails of the loss distribution are more pronounced while in absence of correlation the loss tends to be reduced and to have a concentration around 0. The reduction of the right tail is amplified when correlation is negative. The distribution is asymmetric because of the test setup where losses are favoured with respect to gains.
Figure 3: The loss distribution for different values of rank correlation ($\rho = -1.0, 0.0, 1.0$) is shown where a negative value of the loss is corresponding to a profit for the bank.

From a risk perspective, the cases where high frequency correspond to low RRs ($\rho > 0$) are the most relevant. The impact of rank correlation on the charge distribution is shown in Figure 4 where the DRC increase with respect to the uncorrelated case is presented. As expected, the DRC is a monotonic function of $\rho$ with an increase of 2% for $\rho = 0.2$ and 12% for $\rho = 1.0$

Default risk charge vs. rank correlation

Figure 4: The DRC for various rank correlations normalized by the DRC for $\rho = 0$ is shown.

4 Conclusion

The new requirements for the capitalization of default risk in the Trading Book pose some methodological challenges and demand for a departure from the current IRC model. In particular, the simulation of default times (to properly take into account possible maturity mismatches between a position and
Throughout the appendix, we use the following notation. The set of real numbers is abbreviated by $\mathbb{R}$, the natural numbers with $\mathbb{N}$ resp. $\mathbb{N}_0$ if the zero is included. The flooring of a real number $x$ to the largest natural number smaller than $x$ is written as $\lfloor x \rfloor = \text{argmax} \{ n \leq x | n \in \mathbb{N}_0 \}$.

The indicator function is abbreviated by $\mathbb{1}_A$ for some condition $A$. If the condition $A$ is true, then the function $\mathbb{1}_A$ is evaluated to 1, otherwise it is zero.

A vector $v \in \mathbb{R}^n$ is denoted equivalently as $v = (v(1), ..., v(n)) = (v(i))_{i=1, ..., n} = (v(i))$. Matrices are written in capital letters, its entries in small letters. Hence, a matrix $A \in \mathbb{R}^{n \times m}$ is equivalently written as $A = (a_{ij})_{i=1, ..., n; j=1, ..., m} = (a_{ij})$. The $j$-th column of matrix $A$ is denoted by $a_j = (a_{i,j})_{i=1, ..., n}$ and similarly the $i$-th row as $a_i = (a_{i,j})_{j=1, ..., m}$. The high comma stands for the transposition of a matrix, i.e. $A^\top$ is the transposed matrix $A$.

### Appendix A  Notation

#### Appendix A Notation

Throughout the appendix, we use the following notation. The set of real numbers is abbreviated by $\mathbb{R}$, the natural numbers with $\mathbb{N}$ resp. $\mathbb{N}_0$ if the zero is included. The flooring of a real number $x$ to the largest natural number smaller than $x$ is written as $| x | = \text{argmax} \{ n \leq x | n \in \mathbb{N}_0 \}$.

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### Appendix B  Fundamental Risk Factors

Let $N_{FRF}$ be the number of predefined fundamental risk factors. For $j = 1, ..., N_{FRF}$ let $r_j = (r_{1,j}, ..., r_{T,j}) \in \mathbb{R}^T$ be predefined risk factor time series. We assume that the risk factor time series is ordered such that $r_{t,j}$ represents a more recent point in time than $r_{t+1,j}$. Furthermore, we assume that there are entries for all business days between 1 and $T$.

#### B.1  Risk factor returns and return’s normalization

For all risk factors $j = 1, ..., N_{FRF}$, some period length $\delta \in \mathbb{N}$ and for each index $i = 1, ..., \lfloor (T - 1)/\delta \rfloor$, we calculate non-overlapping returns $t_{i,j} = (t_{1,j}, ..., t_{(T-1)/\delta,j})$ by

$$t_{i,j} = r_{1+(i-1)\delta,j} - r_{1+i\delta,j}.$$  \hspace{1cm} (B.1)

We assume that the returns $t_{i,j}$, $j = 1, ..., N_{FRF}$, have a zero mean. Thus, for all $i = 1, ..., \lfloor (T - 1)/\delta \rfloor$ and $j = 1, ..., N_{FRF}$, the returns are normalized via

$$\check{t}_{i,j} = \frac{t_{i,j}}{\sqrt{\frac{1}{\lfloor (T - 1)/\delta \rfloor} \sum_{k=1}^{\lfloor (T - 1)/\delta \rfloor} t_{k,j}^2}}.$$  \hspace{1cm} (B.2)

For financial time series this assumption is usually more accurate than any estimate of the mean from the data. We define the matrix of normalized returns $\check{r}_t \in \mathbb{R}^{\lfloor (T - 1)/\delta \rfloor \times N_{FRF}}$ with the $j$-th column equal to $\check{r}_{i,j} = \check{t}_{i,j} = (\check{t}_{1,j}, ..., \check{t}_{(T-1)/\delta,j})$ by

$$\check{r}_t = (\check{r}_{1,t}, ..., \check{r}_{N_{FRF},t}).$$  \hspace{1cm} (B.3)

#### B.2  PCA and fundamental risk factors

With the normalized returns $\check{r}_t$, $j = 1, ..., N_{FRF}$, a principal component analysis is performed. To be more precise, we use the singular value decomposition, scale the variables to have unit variance and do not shift the variables to be zero centered.
This gives us a matrix \( V \in \mathbb{R}^{N_{\text{FRF}} \times N_{\text{FRF}}} \) whose columns contain the eigenvectors. The eigenvalue of eigenvector \( j, j = 1, \ldots, N_{\text{FRF}} \), is denoted by \( \lambda_j \). The eigenvectors are transformed to a matrix \( \text{FRF} \in \mathbb{R}^{N_{\text{FRF}} \times \lfloor (T - 1)/\delta \rfloor} \) via

\[
\text{FRF} = \text{diag}\left(\frac{1}{\sqrt{\lambda_1}}, \ldots, \frac{1}{\sqrt{\lambda_{N_{\text{FRF}}}}}\right) \cdot V', \text{fr} \text{fr}
\] (B.4)

From this matrix, we select the first \( Q \in \mathbb{N} \) components as

\[
Q = \arg\min \left\{ j = 1, \ldots, N_{\text{FRF}} \mid \frac{\sum_{i=1}^j \lambda_i}{\sum_{i=1}^{N_{\text{FRF}}} \lambda_i} \geq p \right\}, \quad (B.5)
\]

where \( p \in [0, 1] \) is a chosen threshold. The first \( Q \) rows of the matrix \( \text{FRF} \) are the so-called fundamental risk factors (FRF) used as explanatory variables for the issuers’ time series.

### B.3 Linear regression on fundamental risk factors

Each issuer \( j, j = 1, \ldots, N_{\text{iss}} \), has an associated risk factor time series \( \text{issr}_j = (\text{issr}_{1,j}, \ldots, \text{issr}_{T,j}) \in \mathbb{R}^{T,j} \). The time series is usually a series of credit spreads or natural logarithm of equity prices. As before, for all \( i = 1, \ldots, \lfloor (T_j - 1)/\delta \rfloor \) we first calculate the returns

\[
\text{issrt}_{i,j} = \text{issr}_{1 + (i-1)\delta,j} - \text{issr}_{1+i\delta,j}.
\] (B.6)

Then, for all \( i = 1, \ldots, \lfloor (T_j - 1)/\delta \rfloor \), the returns are normalized according to

\[
\text{issrtn}_{i,j} = \frac{\text{issrt}_{i,j}}{\sqrt{\sum_{k=1}^{\lfloor (T_j - 1)/\delta \rfloor} \text{issrt}_{k,j}^2}}.
\] (B.7)

A linear least square regression with a fixed intercept of zero of the normalized returns to the FRFs determines a coefficient matrix \( (\beta_{jl}) \), \( j = 1, \ldots, N_{\text{iss}}, l = 1, \ldots, Q \).

### Appendix C Simulation of Correlated Default Times

We model the default time \( \tau_j^s \) for issuer \( j \) and scenario \( s \) via an issuer-specific exponential distribution with rate parameter \( \theta_j \) whose cumulative distribution \( F_j \) is given by

\[
F_j(x) = 1 - \exp^{-\theta_j x}, \quad x \geq 0.
\] (C.1)

For each issuer \( j \), the rate parameter \( \theta_j \) is calibrated to the one-year default probability \( PD_j \). Thus, we have to solve \( F_j(1) = PD_j \). This yields to

\[
\theta_j = - \log (1 - PD_j).
\] (C.2)

The default time \( \tau_j^s \) for issuer \( j \) and scenario \( s \) is simulated via the inverse transform sampling\(^3\) which gives

\[
\tau_j^s = \frac{1}{\log (1 - PD_j)} \log (u)
\] (C.3)

where \( u \) is drawn from a uniform distribution \( U \) on \([0;1]\).

We apply a Gaussian copula framework. Therefore, the uniform distribution \( U \) is replaced by the cumulative standard normal distribution \( \Phi \) whose realizations \( z_j^s \) depend on the regression coefficients \( \beta_{jl} \) between issuer \( j \) and FRF \( l \). This yields to

\[
\tau_j^s = \frac{1}{\log (1 - PD_j)} \log (\Phi(z_j^s))
\] (C.4)

\(^3\)This method uses the fact that the distribution of \( F^{-1}(U) \) with \( U \) as a uniform distribution on \([0;1]\) is distributed according to \( F \).
where $z^*_j$ is computed as

$$z^*_j = \sum_{l=1}^{Q} \beta_{jl} \epsilon^*_l + \sqrt{1 - \sum_{l=1}^{Q} \beta^2_{jl} \epsilon^*_j},$$

(C.5)

Here $\epsilon^*_l$, $\epsilon^*_j$ are independent standard normal variables.

In order to move from a Gaussian copula to a Student’s $t$ copula with $k$ degrees of freedom, only two modifications need to be applied to the simulation scheme. First, the variable $z^*_j$ must be replaced by

$$\bar{z}^*_j = z^*_j \sqrt{\frac{k}{\xi(k)^*}},$$

(C.6)

where $\xi(k)^*$ follows a Chi-squared distribution with $k$ degrees of freedom. Second, the cumulative standard normal distribution function $\Phi$ in Eq. C.4 must be replaced by the cumulative Student’s $t$ distribution function with $k$ degrees of freedom.

Appendix D Simulation of Recovery Rates

D.1 Minimum Entropy Principle

The entropy $E$ is a function from the space of non-negative $L^1$-integrable functions to the real numbers. Let $f$ be a density function, $f : \mathbb{R} \rightarrow [0; \infty)$, $\int_0^1 |f(x)|\,dx < \infty$, then the entropy $E(f)$ is defined by

$$E(f) = \int_0^1 f(x) \log(f(x))\,dx.$$  \hspace{1cm} (D.1)

It can be shown, that $E(f)$ is minimal if and only if $f$ is a multiple of the uniform distribution, cf. Proposition 2.8 in Liese and Vajda (1987)$^4$. The uniform distribution can be thought of as a distribution over $[0; 1]$ with no additional information, since every value has the same probability density. Introducing certain constraints on $f$ and minimizing the entropy leads to a function that just fulfills the constraints without adding further unnecessary and unwanted information.

D.2 Constraints

We use this minimum entropy principle in order to find a density function for the simulation of the RRs. We interpret the density function $f$ as the distribution of the RR of the unsecured debt. The unsecured debt is divided into senior unsecured and subordinated. The dependence is defined by a number of constraints that are introduced in the following.

At a certain point $x_0 \in (0; 1)$ of the RR of the unsecured debt, the RR of the senior unsecured is equal to one, i.e. $rr_{sen} = 1$, while the subordinated tranche has not yet received anything, i.e. $rr_{sub} = 0$. For RRs $x \leq x_0$, we have $rr_{sen} = x/x_0$ and $rr_{sub} = 0$. For RRs $x > x_0$, we have $rr_{sen} = 1$, while $rr_{sub} = (x - x_0)/(1 - x_0)$. From this we can state the following constraints on $f$, relating the distribution means to the expected values of the respective RRs, namely

$$\int_0^{x_0} \frac{x}{x_0} \cdot f(x)\,dx + \int_{x_0}^1 1 \cdot f(x)\,dx - RR_{sen} = 0$$  \hspace{1cm} (D.2)

$$\Leftrightarrow \int_0^{x_0} xf(x)\,dx + \int_{x_0}^1 x_0 f(x)\,dx - x_0 RR_{sen} = 0$$  \hspace{1cm} (D.3)

$^4$In Physics, the entropy is usually written with the opposite sign of the integral and then maximized. Here, we follow the information theory approach. Moreover, the entropy defined in equation D.1 can be understood as the Kullback Leibler divergence to the uniform distribution.
and
\[
\int_0^{x_0} 0 \cdot f(x)dx + \int_{x_0}^1 \frac{x-x_0}{1-x_0} \cdot f(x)dx - RR_{ab} = 0 \tag{D.4}
\]
\[
\Leftrightarrow \int_{x_0}^1 (x-x_0)f(x)dx - (1-x_0)RR_{ab} = 0. \tag{D.5}
\]

Finally, we also impose a normalization constraint on the function \(f\), i.e.
\[
\int_0^1 f(x)dx - 1 = 0. \tag{D.6}
\]

### D.3 Optimization

For the optimization, we use the Lagrange ansatz. Our Lagrange function \(L\) reads as
\[
L(f, x_0, \lambda_1, \lambda_2, \lambda_3) = E(f) - \lambda_1 \left( \int_0^1 f(x)dx - 1 \right) \tag{D.7}
\]
\[
- \lambda_2 \left( \int_0^{x_0} xf(x)dx + \int_{x_0}^1 x_0f(x)dx - x_0RR_{sm} \right) \tag{D.8}
\]
\[
- \lambda_3 \left( \int_{x_0}^1 (x-x_0)f(x)dx - (1-x_0)RR_{ab} \right). \tag{D.9}
\]

Differentiating \(L\) by \(f(x)\) gives
\[
\frac{\partial L}{\partial f(x)} = \begin{cases} 
\log(f(x)) + 1 - \lambda_1 - \lambda_2 x & \text{if } x \leq x_0, \\
\log(f(x)) + 1 - \lambda_1 - \lambda_2 x_0 - \lambda_3(x-x_0) & \text{if } x > x_0.
\end{cases} \tag{D.10}
\]

Setting this partial differential to zero, we see that
\[
f(x) = \begin{cases} 
Ce^{-\lambda_2 x} & \text{if } x \leq x_0, \\
Ce^{-\lambda_2 x_0+\lambda_3(x-x_0)} & \text{if } x > x_0.
\end{cases} \tag{D.11}
\]

with \(C = e^{\lambda_1-1}\). Obviously, the function \(f\) is continuous in \(x_0\). The variable \(\lambda_1\) is used as the normalization to a density function.

Since \(x_0\) is a variable as well, we have to calculate the partial differential by \(x_0\), that gives
\[
\frac{\partial L}{\partial x_0} = - \lambda_2 \left( x_0f(x_0) + \left( \int_{x_0}^1 f(x)dx - x_0f(x_0) \right) - RR_{sm} \right) \tag{D.12}
\]
\[
- \lambda_3 \left( -x_0f(x_0) - \int_{x_0}^1 f(x)dx + x_0f(x_0) + RR_{ab} \right) \tag{D.13}
\]
\[
= - \lambda_2 \left( \int_{x_0}^1 f(x)dx - RR_{sm} \right) - \lambda_3 \left( - \int_{x_0}^1 f(x)dx + RR_{ab} \right). \tag{D.14}
\]

Again, setting this partial differential to zero, we have
\[
\lambda_2 \left( \int_{x_0}^1 f(x)dx - RR_{sm} \right) = \lambda_3 \left( \int_{x_0}^1 f(x)dx - RR_{ab} \right). \tag{D.15}
\]

Solving equation D.2 by \(RR_{sm}\) and equation D.4 by \(RR_{ab}\) and plugging both into equation D.15 gives
\[
\lambda_2 \left( \int_{x_0}^1 f(x)dx - \int_0^{x_0} \frac{x}{x_0} f(x)dx - \int_{x_0}^1 f(x)dx \right) = \lambda_3 \left( \int_{x_0}^1 f(x)dx - \int_{x_0}^1 \frac{x-x_0}{1-x_0} f(x)dx \right) \tag{D.16}
\]
\[
\Leftrightarrow \lambda_2 \left( - \int_0^{x_0} \frac{x}{x_0} f(x)dx \right) = \lambda_3 \left( \int_{x_0}^1 \frac{1-x}{1-x_0} f(x)dx \right) \tag{D.17}
\]
\[
\Leftrightarrow \lambda_2(1-x_0) \int_0^{x_0} xf(x)dx = \lambda_3 x_0 \int_{x_0}^1 (x-1)f(x)dx. \tag{D.18}
\]
Inserting the exponential form of \( f \) from equation D.11 we obtain
\[
\lambda_2 (1 - x_0) C \int_0^{x_0} x e^{\lambda_2 x} dx = \lambda_3 x_0 C \int_{x_0}^1 (x - 1) e^{\lambda_2 x_0 + \lambda_3 (x - x_0)} dx.
\tag{D.19}
\]
Canceling out \( C \) and substituting \( y = x / x_0 \) on the left side and \( y = (1 - x) / (1 - x_0) \) on the right side, it holds
\[
\lambda_2 (1 - x_0) x_0 \int_0^1 ye^{\lambda_2 y_0} dy = \lambda_3 x_0 (1 - x_0) (x_0 - 1) \int_0^1 ye^{\lambda_2 x_0 + \lambda_3 (x_0 - 1) y + \lambda_3 (1 - x_0)} dy
\tag{D.20}
\]
\[\Leftrightarrow\]
\[
\lambda_2 x_0 \int_0^1 ye^{\lambda_2 y_0} dy = \lambda_3 (x_0 - 1) e^{\lambda_2 x_0 + \lambda_3 (1 - x_0)} \int_0^1 ye^{\lambda_3 (x_0 - 1) y} dy.
\tag{D.21}
\]
Obviously, a solution is
\[
\lambda_2 x_0 = \lambda_3 (x_0 - 1).
\tag{D.22}
\]

### D.4 Conditional distributions

We derive two conditional distributions from the density \( f \) of equation D.11. First, the distribution of the senior unsecured \( RR \) given it is smaller than 1 and, second, the distribution of the subordinated \( RR \) given it is greater than 0. To do so, each part of the function \( f \) has to be positive linear scaled to be a distribution on \([0; 1]\). Thus, for the conditional distribution for the senior unsecured \( RR \), we have to scale with \( x / x_0 \) while for the subordinated \( RR \), we have to scale with \((x - x_0) / (1 - x_0)\). This leads to the growth parameter of \( x_0 \lambda_2 \) for the senior unsecured and \( \lambda_3 (1 - x_0) \) for the subordinated tranche. To see that, substitute \( y = x / x_0 \) in \( f \) for \( x \leq x_0 \) and substitute \( y = (x - x_0) / (1 - x_0) \) in \( f \) for \( x > x_0 \), similar to equation D.20.

Applying equation D.22, we see that the growth parameter of the two conditional distributions have the same absolute value but different signs. Exponential distributions on \([0; 1]\) with growth parameter \( \lambda \) have the density
\[
\frac{\lambda}{e^\lambda - 1} e^{\lambda x}.
\tag{D.23}
\]
The mean of an exponential distribution with growth parameter \( \lambda \) is one minus the mean of an exponential distribution with growth parameter \(-\lambda\), as seen by
\[
\int_0^1 x \frac{\lambda}{e^\lambda - 1} e^{\lambda x} dx = \frac{1}{1 - e^{-\lambda}} \frac{1}{\lambda} \tag{D.24}
\]
\[= 1 - \left( \frac{1}{1 - e^{-(-\lambda)}} - \frac{1}{-\lambda} \right) \tag{D.25}
\]
\[= 1 - \left( \frac{1}{1 - e^{-\lambda}} - \frac{1}{-\lambda} \right) \tag{D.26}
\]
Hence, if the mean of the conditional distribution of the senior unsecured \( RR \) is denoted by \( \mu \), then the mean of the conditional distribution of the subordinated \( RR \) is \( 1 - \mu \). With the notation from the beginning of the Section 2.3, the probability of the condition \( x > x_0 \) is abbreviated by
\[
p = \int_{x_0}^1 f(x) dx.
\tag{D.27}
\]
Applying the equations D.2 and D.4, we have
\[
RR_{ab} = (1 - \mu) \cdot p + 0 \cdot (1 - p), \tag{D.28}
\]
\[
RR_{sn} = 1 \cdot p + (1 - p) \cdot \mu. \tag{D.29}
\]
These equations can be solved for \( \mu \) and \( p \) which yields to
\[
p = \frac{RR_{ab}}{RR_{ab} - RR_{sn} + 1}, \tag{D.30}
\]
\[
\mu = RR_{sn} - RR_{ab}. \tag{D.31}
\]
The equation D.30 is the one needed in Section 2.3 to solve for the four variables defined therein.
Appendix E  Correlation between Default Times and Recovery Rates

For issuer $j$ in scenario $s$ a default is registered if and only if its default time $\tau^s_j < 1$. The scenarios $s = 1, \ldots, N_{\text{scen}}$ are ordered such that the number of defaults per scenario over all issuers is in decreasing order, i.e.

$$\sum_{j=1}^{N_{\text{iss}}} \mathbb{1}_{\{\tau^s_j < 1\}} \geq \sum_{j=1}^{N_{\text{iss}}} \mathbb{1}_{\{\tau^t_j < 1\}} \quad \forall s \leq t.$$  (E.1)

Assume we have $n_j$ default scenarios for issuer $j$, then we define

$$n_j(i) = n_j^{\text{ab}}(i) + n_j^{\text{sr}}(i)$$  (E.2)

for all $i = 1, \ldots, n_j$, where $n_j^{\text{ab}}(i)$ is the $i$-th entry of the vector $n_j^{\text{ab}} \in \mathbb{R}^{n_j}$. The order of the RRs $n_j$ is determined and saved in a variable $\text{order}^{n_j} = (\text{order}^{n_j}_1, \ldots, \text{order}^{n_j}_{n_j})$ such that

$$n_j(\text{order}^{n_j}_k) \geq n_j(\text{order}^{n_j}_l) \quad \forall k \leq l.$$  (E.3)

Now we generate for each issuer $j$ a random correlated order map $d^i_j$, $i = 1, \ldots, n_j$, by

$$d^i_j = \rho \Phi^{-1} \left( \frac{i - 0.5}{n_j} \right) + \sqrt{1 - \rho^2} b_i$$  (E.4)

where $b_i, i = 1, \ldots, n_j$, are independently drawn from standard normal distribution, and $\Phi$ is the cumulative distribution function of the standard normal distribution. The order of the order map $d^i_j$ is determined and saved in a variable $\text{order}^{d^i_j} = (\text{order}^{d^i_j}_1, \ldots, \text{order}^{d^i_j}_{n_j})$ such that

$$d^i_j(\text{order}^{d^i_j}_k) \geq d^i_j(\text{order}^{d^i_j}_l) \quad \forall k \leq l.$$  (E.5)

The RRs are then reshuffled and saved in the variable $\tilde{n}_j^{\text{sr}}$ according to

$$\tilde{n}_j^{\text{sr}}(i) = n_j^{\text{s}}(\text{order}^{\text{order}^{d^i_j}}_k), \quad \text{s} \in \{\text{sec}, \text{sen}, \text{sub}\}.$$  (E.6)

5 Declarations of Interest

All authors are employed by UniCredit SpA or its legal entities.

References


UniCredit Foundation
Piazza Gae Aulenti, 3
UniCredit Tower A
20154 Milan
Italy

Giannantonio De Roni – Secretary General
e-mail: giannantonio.deroni@unicredit.eu

Annalisa Aleati - Scientific Director
e-mail: annalisa.aleati@unicredit.eu

Info at:
www.unicreditfoundation.org